On the hat problem, its variations, and their applications

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Abstract

The topic of our paper is the hat problem in which each of n players is randomly fitted with a blue or red hat. Then everybody can try to guess simultaneously his own hat color by looking at the hat colors of the other players. The team wins if at least one player guesses his hat color correctly, and no one guesses his hat color wrong; otherwise the team loses. The aim is to maximize the probability of a win. There are known many variations of the hat problem. In this paper we give a comprehensive list of variations considered in the literature. We describe the applications of the hat problem and its variations, and their connections to different areas of science. We give the full bibliography of any papers, books, and electronic publications about the hat problem.

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1 Introduction

In the hat problem, a team of n players enters a room and a blue or red hat is randomly placed on the head of each player. Each player can see the hats of all of the other players but not his own. No communication of any sort is allowed, except for an initial strategy session before the game begins. Once they have had a chance to look at the other hats, each player must simultaneously guess the color of his own hat or pass. The team wins if at least one player guesses his hat color correctly and no one guesses his hat color wrong; otherwise the team loses. The aim is to maximize the probability of a win.

The hat problem with seven players, called the "seven prisoners puzzle", was formulated by T. Ebert in his Ph.D. Thesis [20]. The hat problem was also the subject of articles in The New York Times [46], Die Zeit [9], and abcNews [44]. It is also a one of subjects of the webpage [7].

The hat problem with $2^k - 1$ players was solved in [22], and for 2^k players in [17]. The problem with n players was investigated in [11]. The hat problem and Hamming codes were the subject of [12].

There are known many variations of the hat problem. For example the generalized hat problem with n players and q colors was investigated in [40]. In the papers [1, 15, 35] there was considered a variation in which passing is not allowed, thus everybody has to guess his hat color. The aim is to maximize the number of correct guesses. The authors of [25] investigated several variations of the hat problem in which the aim is to design a strategy guaranteeing desired number of correct guesses. In [30] there was considered a variation in which the probabilities of getting hats of each colors do not have to be equal. The authors of [5] investigated a problem similar to the hat problem. There are n players which have random bits on foreheads, and they have to vote on the parity of the n bits. The hat problem on a graph is as follows. There is a graph, where vertices correspond to players and a player can see each player to whom he is connected by an edge. This variation of the hat problem was first considered in [38]. There were proven some general theorems about the hat problem on a graph, and the problem was solved on trees. Additionally, there was considered the hat problem on a graph such that the only known information are degrees of vertices. In [39] the problem was solved on the cycle C_4. Further results about the hat problem on a graph were established by Uriel Feige [24]. For example, there the problem was solved for bipartite graphs, and planar graphs containing a triangle. Based on these and some other results, the author conjectured that for every graph there is an optimal strategy in which all vertices who do not belong to the maximum clique always pass.

The hat problem and its variations have many applications and connections to different areas of science, for example: information technology [8], linear programming [25], genetic programming [14], economy [1, 35], biology [30], approximating Boolean functions [5], and autoreducibility of random sequences [6, 20-23].

In this paper we give a comprehensive list of variations of the hat problem considered in the literature. We also present what is already known about each variation. For some variations we give a strategy which solves the problem. Next we describe the applications of the hat problem and its variations, and their connections to different areas of science. We give the full bibliography of any papers, books, and electronic publications about the hat problem.

2 Applications of the hat problem

In this section we present applications of the hat problem and its variations. We also consider their connections to different areas of science.

Information technology. The paper [8] shows the strong connection between the hat problem and the following problem. In storing or transmitting digital data, there is always some risk of distortion: a 0 might accidentally flip to 1 or vice versa. One way to deal with this problem is to introduce some redundancy into the transmission – for instance, by sending each bit multiple times. However, transmitting too many extra bits is costly and ineffective. We need to protect k bits of data against the possibility of an error by using the minimal number of additional "check bits". Note that the method must not only detect the error, but also determine its precise location, so that the user can recover the original message every time. This problem has been solved using Hamming codes – codes which detect and correct errors. So called covering codes are strongly related to Hamming codes. The website [41] contains up-to-date data on the best known covering codes. The coding theory (for more information, see [47]) was inaugurated by Richard Hamming. He realized that there is a way to use as few bits as possible and still receive the correct message, but he was unable to explicitly prove it [42]. The work of Hamming piqued the interest of other mathematicians, such as Claude Shannon who worked on the information theory aspects of coding to achieve clear data transmission. Some of work of Shannon provides us with high sound quality of compact discs. Even though compact discs may have visible scratches and thumb prints, a compact disc player still reads the song accurately. This is because of the error-correcting capabilities built into the compact discs. The hat problem with $2^k - 1$ or 2^k players has been solved using the Hamming codes. The hat problem with $n \notin \{2^k - 1, 2^k\}$ players, and the generalized hat problem with any number of players and at least three colors are unsolved. These hat problems may have further connections to and applications in information technology.

Genetic programming. In [14] the authors try to solve the hat problem with $n \notin \{2^k - 1, 2^k\}$ players using genetic programming. The aim is not only to solve the hat problem, but also to learn the way in which the genetic programming works, and what is its effectiveness, because the hat problem seems to be a typical one to solve using genetic programming. As a result it can help us in solving another, even practical problems using genetic programming.

Biology. In [30] it is shown that one of most important problems in cell biology is to understand functionality of each and every gene of any living organism. A mammoth project, called the Deletion Project, is underway to study the DNA of the yeast organism. The genome of yeast organism has been completely mapped out. It has about 6000 genes. Experiments on yeast cells, under the project, have the following basic operations:

1. removal of a gene from the cell;
2. placement of the cell in a chamber at a set temperature;
3. examination of every one of the remaining cells to determine whether or not it is active.

The data vector generated is of order 1 x 6000. Every entry in the vector, except one, is 0 (inactive) or 1 (active). The missing entry corresponds to the deleted gene. Steps 1, 2, and 3 should be repeated with respect to every gene. Thus, at the set temperature, we will have 6000 binary data vectors, each vector having exactly one blank space. The whole cell is also placed in the chamber without removing any of its genes. The data vector generated will not have any blanks. Using all these data vectors, one has to guess what would have been the role of the deleted gene had it been present in the cell. It can be hoped that the hat problem might have some pointers.

Mathematics: the autoreducibility of random sequences. In the Ph.D. Thesis of Todd Ebert [20] and in [23] it can be read that the autoreducibility of random sequences is the problem of deducing a property of a random binary sequence when some of the bits of the sequence upon which the property depends are not known. This occurs quite often in practice when, due to time and other resource constrains, a decision is made using only partial information. This consideration is closely related to complexity theory since a decision must be made before a limited resource such as time has been exhausted. In [22, 23] the authors use the hat problem to investigate the autoreducibility of random sequences. The problem of autoreducibility of random sets, which is strongly connected to the problem of autoreducibility of random sequences, was investigated in [6, 21].

Cellular automata. It can be seen that a similarity exists between the hat problem on a graph and so called cellular automata.

First, let us consider asynchronus threshold networks studied by Noga Alon in [2]. There is a graph G with an initial sign $s(v) \in \{-1, 1\}$ for every vertex v. When v becomes active, it changes its sign to $s'(v)$ which is the sign of majority of its neighbors (we define $s'(v) = 1$ if there is a tie). We say that G is in a stable state if $s(v) = s'(v)$ for every vertex v. The timing is synchronus if all vertices become active simultaneously. The timing is asynchronus when only one vertex becomes active at a time. Alon has proven that for every threshold network with all positive edge weights there is an asynchronus run with at most one sign change per vertex which leads the network to a stable state.

The problem above is connected to societies with symmetric influences introduced by Svatjopluk Poljak and Miroslav Sura [43]. The authors proposed a simple model of society with a symmetric function $w(u, v)$ measuring the influence of the opinion of member v on that of member u. The opinions are chosen from the set $\{0, 1, \ldots, p\}$ for some positive integer p. At each step everyone accepts the majority opinion (with respect to w) of the other members (if there are two or more majority opinions, then he accepts the highest one). Obviously, the behavior of such a society is periodic after some initial time. The authors have proven that the length of the period is either one or two. They also concluded that if the influence function w is not symmetric, then the period can be arbitrarily large.

Another model of social influences was introduced by French [26] and Harary [31]. The main differences between their model and the one of Poljak and Sura are that the "opinions" of the members $u \in V$ are real numbers, influences $w(u, v)$ between members are nonnegative real numbers, and the opinion of a member u is the average opinion of the others. For a survey on this topic, see the book [45].

For more information about cellular automata, see [18].

From now to the end of this section we consider variations of the hat problem.

Linear programming One of the theorems about the hat problem proved in [25] can be represented as a special case of the well known fact that linear programs with integer constraints and a totally unimodular constraint matrix always have integer optimal solutions. The connection between total unimodularity and the solution of integer programs was apparently first shown in [34]. It can be hoped that the hat problem has further connection to and application in linear programming.

Economy Nicole Immorlica in her Ph.D. Thesis [35] and the authors of [1] project auctions in which the aim is to maximize the profit of the seller. During investigating this problem, they consider a variation of the hat problem in which everybody has to guess his hat color and we are interested in guaranteeing as much correct guesses as possible. This problem is related to the auction problem as follows. Consider the case where are only two types of bidders, those with high valuation for the item, h; and those with a low valuation for the item, l. Mapping h to the color red and l to the color blue, a solution of the hat problem would offer half of the h bids at a price h and half of the l bids at a price l, thus the profit of such an auction would be at least half of optimal revenue.

Mathematics: approximating a Boolean function The authors of [5] consider the problem of approximating a Boolean function $f\colon \{0,1\}^n \to \{0,1\}$ by the sign of an integer polynomial p of degree k. We say that a polynomial $p(x)$ predicts the value of $f(x)$ if, whenever $p(x) \geq 0$, $f(x) = 1$, and whenever $p(x) < 0$, $f(x) = 0$. A low-degree polynomial p is a good approximator for f if it predicts f at almost all points. Given a positive k, and a Boolean function f, the problem is how good is the best degree k approximator to f. To investigate this problem, the authors use the problem similar to the hat problem in which every one from an odd number of players has 0 or 1 on his forehead. Everybody has to guess the parity of the bits. The game is won if more than half of all guesses are correct.

3 Variations of hat problem

Now, we give a comprehensive list of variations of the hat problem considered in literature. We also present what is already known about each variation. For some variations we give a strategy which solves the problem.

(1) "The generalized hat problem with n players and q colors" was first investigated in [40]. Every one of n players has got a hat of one from q possible colors, and the probabilities of getting hats of all colors are equal. We say that a strategy is symmetric if every player makes his decision on the basis of only numbers of hats of each color seen by him, and all players behave in the same way. A strategy is nonsymmetric if it is not symmetric. The authors of [30] solved the hat problem with three players and three colors by giving a symmetric strategy found by computer, and proving that it is optimal. In [37] the problem was solved by proving the optimality of a nonsymmetric strategy found without using computer. There were also proven some upper bounds on the effectiveness of any strategy for the generalized hat problem with n players and q colors. Additionally, there were considered the numbers of strategies that suffice to be verified to solve the hat problem, or the generalized hat problem. N. Alon [3] proved a lower bound on the maximum chance of success for the generalized hat problem.

(2) There are n players and two colors. Everybody has to guess his hat color. The aim is to find a strategy guaranteeing as many correct guesses as possible. It is known that guaranteeing $\lfloor n/2 \rfloor$ correct guesses is the best possible. The following strategy is optimal. Have players paired up. If the number of players is odd, then the unpaired one always guesses he has, let us say, a blue hat. In each pair one player guesses he has a hat of the same color as the other player, while the other player guesses he has a hat of the color another than the first player, see [13, 15, 32, 49, 50].

(3) It differs from the previous problem only in that there are $q \geq 3$ colors. It has been proven that guaranteeing $\lfloor n/q \rfloor$ correct guesses is the best possible. The following strategy is optimal. Number players 1 to n, and colors 1 to q. The ith player guesses as if the sum of colors of all hats (including own) is congruent to i modulo q, see [15].

(4) It differs from the previous problem only in that there is a directed graph G determining players seen by each player – if there is an arc from u to v, then the player u can see the player v. Optimal strategy for this problem is not known.

There exist some lower and upper bounds on the number $t(G)$ which means the maximum number of correct guesses that can be guaranteed. For a directed graph G, let $c(G)$ denote the maximal number of vertex-disjoint cycles in G, and let $F(G)$ denote the minimum number of vertices whose removal from G makes the graph acyclic. Then $c(G) \leq t(G) \leq F(G)$, see [15].

(5) It differs from the previous problem only in that there is also a graph H determining each player to guess the hat color of the particular player (possibly own) – if there is an arc from u to v, then the player u has to guess the hat color of the player v. Let $t_q(G, H)$ mean the maximum number of correct guesses that can be guaranteed when there are q colors. There is known only the fact that $t_q(G, H) > 0$ if and only if there is a vertex of H whose outdegree is greater than 1, or there is a directed cycle in the union of G and H, see [15].

(6) It differs from the previous problem only in that there are $a_1, a_2, \ldots,$ a_q hats of the color $1, 2, \ldots, q$, respectively. There are few facts known for the variation, one of them is as follows. By $t(n; a_1, a_2, \ldots, a_q)$ let us denote the maximum number of correct guesses that can be guaranteed when there are n players, and a_1 hats of the first color, a_2 hats of the second color, and so on up through a_q hats of qth color. Of course, we need $a_1 + a_2 + \ldots + a_q \geq n$ to ensure that we have enough hats. Without loss of generality we may assume that $0 < a_i \leq n$, for all i. It is easy to notice that if $a_1 + a_2 + \ldots + a_q = n$, then $t(n; a_1, a_2, \ldots, a_q) = n$, see [15].

(7) There are n players standing in a line and two colors. Everybody can see the hat colors of players before him, but neither his nor those of players behind him. Players have to guess their hat colors sequentially, starting from the back of the line. Everybody can hear the answer called out by each player. We are interested in a strategy guaranteeing as many correct guesses as possible. The following strategy is optimal. If the last player sees an odd number of red hats in front of him, then he guesses he has a red hat. Otherwise he guesses he has a blue hat. Player $n-1$ will deduce his own hat color from the information said by the last player. Similar reasoning applies to each player going up the line. Player i sums the number of red hats he sees and red guesses he hears. If the sum is odd, then he guesses he has a red hat. Otherwise he guesses he has a blue hat. Of course, it is not possible to guarantee the correctness of the guess of the player who guesses as first, thus guaranteeing $n-1$ correct guesses is the best possible, see [4, 19, 27, 49].

(8) It differs from the previous problem only in that there are $q \geq 3$ colors.

Now also the maximum number of correct guesses that can be guaranteed is $n-1$. By $v_1, v_2, \ldots, v_n$ let us denote players, and by $1, 2, \ldots, q$ let us denote colors. Let y_i represent the hat color of player v_i, and let us define $Y_i = \sum_{j=i}^{n} y_j \mod q$. The following strategy is optimal. Player v_1 guesses he has a hat of the color $Y_2 = \sum_{i=2}^{n} y_i \mod q$. For each $i > 1$ player v_i can see the values $y_{i+1}, \ldots, y_n$, and has heard the values Y_2 and $y_2, \ldots, y_{i-1}$. As an effect, he solves the expression for Y_2 to get y_i. As the result, $n-1$ players guess their hat colors correctly, see [4, 19].

(9) It differs from the two previous problems only in that the seeing radius and/or the hearing radius are limited (there are $q \geq 2$ colors). The seeing radius of a player is the maximum number of players that he can see ahead of him. The hearing radius of a player is the maximum number of players ahead of him that can hear him. We assume that the seeing (hearing, respectively) radius is the same for all players, and we denote it by s (h, respectively). For this variation it is known only that the maximum number of correct guesses that can be guaranteed is $n - \lceil n/(\min(s,h)+1) \rceil$, see [4].

(10) There are n players and two colors. There is also a clock and as every minute elapses, everybody can guess his hat color. Time elapses after n minutes, and everybody who has not tried to guess his hat color loses. If some player guesses his hat color wrong, then all players lose. Is there a strategy such that everybody wins? No, although we can try to find a strategy such that as many players as possible wins, see [27].

(11) It differs from the previous problem only in that there is an additional player who comes to the team and says "somebody has a blue hat" or "everybody has a red hat" or something else. Does it can help to guarantee that everybody will win? Assume that the additional player says that somebody has a blue hat. Let us consider the following strategy. Everybody counts blue hats he sees. After k minutes, if nobody has tried to guess his hat color, then everybody who sees $k-1$ red hats guesses he has a red hat. If at least two players have a red hat, then the information from the additional player that somebody has a red hat is a fact known by everybody. Paradoxically, it has a value. The information from the additional player is called common knowledge. That is, everybody knows it, and everybody knows that everybody knows it, and everybody knows that everybody knows that everybody knows it, etc. Players can use this meta-information to derive their own hat colors, see [10, 27].

(12) There are three players, A, B, and C. There are four green and four

red stamps. Players are blindfolded, and two stamps are pasted on the head of each player. After removing the blindfolds, A, B, and C are asked in turn about colors of own stamps. No player knows the answer. Now A is asked once more. He again does not know the answer. Now B is asked, and he replies "yes". What are the colors of the stamps of B? The answer is that he has one green, and one red stamp, see [29].

(13) There are three players and two colors. Everybody has to simultaneously guess his hat color or pass. The team wins if at least one player guesses his hat color correctly and nobody guesses his hat color wrong. The probabilities of the eight cases which can appear does not have to be the same. How does it influence the strategy which should be applied by the team? It has been proven (using computer) that to solve the problem it suffices to calculate the chance of success for a family of twelve strategies, see [30].

(14) It differs from the previous variation only in that there are n players and $q \geq 2$ colors, see [40].

(15) In the "Gabay – O'Connor hat problem" there are an infinite number of players numbered $1, 2, \ldots,$ and two colors. Everybody has to guess his hat color. The team wins if only finite number of guesses are wrong. Is there a strategy guaranteeing that the team will win? Yes, but only if the Axiom of Choice holds, see [32, 33, 51].

(16) The variation called "All right or all wrong" differs from the previous problem only in that the team wins if and only if all guesses are correct or all guesses are wrong. Similarly as for the previous variation, the win of the team can be guaranteed if and only if the Axiom of Choice holds, see [51].

(17) There are ten players and every one of them has a digit from 0 to 9 written on the forehead. Everybody has to guess his digit. The team wins if at least one player does it correctly. The aim is to find a strategy guaranteeing that the team will win. Let us consider the following strategy. Number players 0 to $n-1$. Let s be the sum of the numbers on the foreheads of all players, modulo n. Now let player k guess that $s = k$, that is, guess that his own number is k minus the sum of the numbers he sees, modulo n. This will ensure that player s will be correct, see [51].

(18) The variation called "The color-blind prisoner" differs from the previous problem in that numbers are written in red, one player has a green skin, and one another player does not distinguish green and red. Thus he decides about his guess on the basis of only eight digits. Now it is not possible to guarantee that

the team will win, see [51].

(19) In the variation called "Numbers and hats" there are n players, and every one of them has a distinct real number written on the forehead. Everybody has to choose a blue or red hat for himself. The aim is for the hat colors to alternate in the order determined by the real numbers. There is a strategy guaranteeing that the team will win, but it is very long and complicated, see [51].

(20) In the "Voting puzzle 1" there are an odd number of players, say n. Every one of them has a random bit written on the forehead. Players have to vote on the parity of the bits (by voting 0 or 1).The result of the voting is the bit chosen more often. Players win if the result of the voting is equal to the parity of the bits. The aim is to maximize the chance of success. Optimal strategy gives the chance of success equaling $n/(n+1)$. For the strategy, see [5].

(21) The "Voting puzzle 2" differs from the previous problem only in that everybody can make as many votes as he wants. Optimal strategy gives the chance of success equaling $(2^n-1)/2^n$. For the strategy, see [5].

(22) The "Voting puzzle 3" is as follows. Let S be a set of randomly chosen n bits. There are $\binom{n}{k}$ players, every one of them can see another k-element subset of S. Players participate in a voting, the result of which should be the parity of the bits. Everybody has to make an integer number of votes. If their sum is positive, then the result of the voting is 0. If it is negative, then the result is 1. If the sum is zero, then the result of the voting is not defined. A strategy, based on approximating a Boolean function, guarantees that the team will win, see [5].

(23) In the variation called "Not distinguishable players" there are n players and $q \geq 2$ colors. Every player can see everybody excluding him, but cannot distinguish them. Thus everybody makes his guess on the basis of only numbers of hats of each color seen by him. Every player guesses his hat color or passes. The team wins if at least one player guesses his hat color correctly and nobody guesses his hat color wrong. It has been proven that for large n the maximum chance of success is approximately $(1+(1/3)^{q-1})/2$, for details see [28].

(24) It differs from the previous variation only in that all players have to behave in the same way, see [40].

(25) The variation called "Players do not distinguish colors 1" is as follows. There are n color-blind players and two colors. Before fitting players with hats somebody says players what will be the probability of getting a blue hat, and what of a red hat. By q let us denote the probability of getting a blue hat. It is known that for large n the maximum chance of success is approximately

$(1-q)^{(1-q)/q} - (1-q)^{1/q}$, see [28].

(26) The variation "Players do not distinguish colors 2" differs from the previous problem only in that later (after fitting with hats) somebody says what was the probability of getting a blue hat, and what of a red hat (somebody says how many blue and how many red hats were placed). It is known that, comparing to the previous variation, it does not change the chance of success of optimal strategy, see [28].

(27) In the variation called "Crowns of the Minotaur" there are three players and every one of them is fitted by the Minotaur with a blue or red crown. Every player bets zero or more points on guessing his crown color. A player wins or loses as many points as he has bet, depending on the accuracy of his guess. Then the won and the lost points are added separately, and the team wins if there are more won than lost points. It is known that the maximum chance of success is equal to 7/8. The following strategy is optimal. At first, number players who is first, second, and third. The first player bets one point for red. If the second player sees that the first has a blue crown, then he bets two points for red, otherwise passes. If the third player sees that the first two have both blue crowns, then he bets four points for red, otherwise passes. Unless every player has a blue crown (chance 1/8), everybody wins, see [48].

(28) In "The discarded hat variation" there are $4k-1$ players, and $2k$ blue and $2k$ red hats. Every player is fitted with a hat, and one hat is taken away. Then everybody has to guess his hat color. The aim is to guarantee as many correct guesses as possible. It is known that guaranteeing $3k-1$ correct guesses is the best possible. For an optimal strategy, involving cyclic arrangement of players, see [25].

(29) In "The everywhere balanced variation" there are n players and $q \geq 2$ colors. Let $\{c_1, c_2, \ldots, c_q\}$ be the set of colors, and let H_i mean the set of players having a hat of color c_i. Nobody knows neither to which set he belongs nor what are the cardinalities of sets H_i. The aim is to find a strategy guaranteeing that in every set H_i the number of players guessing their hat colors correctly is between $\lfloor |H_i|/q \rfloor$ and $\lceil |H_i|/q \rceil$. For such strategy (a complicated one), see [25].

(30) The variation "Hat problem on a directed graph asking for at least one correct guess" is as follows. There are n players and two colors. We have a directed graph G determining players seen by each player – if there is an arc from u to v, then the player u can see the player v. What subgraph has to have the visibility graph to ensure the existence of a strategy guaranteeing at least one

correct guess? It has to have a cycle as a subgraph, for details see [32].

(31) It differs from the previous problem in that there are n players and n colors. It is known that now the visibility graph has to be complete, see [32].

(32) It differs from the two previous problems in that there are n players and q colors. What is the maximum number of correct guesses that can be guaranteed? The answer is $\lfloor n/q \rfloor$, see [32].

(33) There are n players and $q \geq 2$ colors. Players are allowed more than one round in which to guess their hat colors. During each round everybody must simultaneously say "My hat color is i", "My hat color is not i", or "Pass", where i is one of the colors. However, if everybody passes in any round, then the team loses. The rounds continue, with each player making a guess or passing, as long as no incorrect guess is made and at least one player guesses his hat color correctly. Then the team wins. It has been proven that the maximum chance of success is $n(q-1)/(1+n(q-1))$, see [16].

(34) In the variation called "Zero-information strategies" there are n players and two colors. Everybody has to simultaneously guess his hat color or pass. The team wins if at least one player guesses his hat color correctly and nobody guesses his hat color wrong. Every player makes his decision without access to any information. Now a winning probability of $1/4$ is asymptotically attainable and optimal, see [40].

(35) "The hat problem on a graph" is as follows. There is a graph, where vertices correspond to players and a player can see each player to whom he is connected by an edge. This variation of the hat problem was first considered in [38]. There were proven some general theorems about the hat problem on a graph, and the problem was solved on trees. Additionally, there was considered the hat problem on a graph such that the only known information are degrees of vertices. In [39] the problem was solved on the cycle C_4. Further results about the hat problem on a graph were established by Uriel Feige [24]. For example, there the problem was solved for bipartite graphs, and planar graphs containing a triangle. Based on these and some other results, the author conjectured that for every graph there is an optimal strategy in which all vertices who do not belong to the maximum clique always pass.

(36) "The modified hat problem" is as follows. There are $n \geq 3$ players. Everyone of them is randomly fitted with a blue or red hat. Players do not have to guess their hat colors simultaneously. In this variation of the hat problem players guess their hat colors by coming to the basket and throwing the proper

card into it. Every player has got two cards with his name and the sentence "I have got a blue hat" or "I have got a red hat". If someone wants to resign from answering, then he does not do anything. The problem was investigated in [36]. There was given an optimal strategy for the problem which always succeeds.

References

[1] G. Aggarwal, A. Fiat, A. Goldberg, J. Hartline, N. Immorlica, and M. Sudan, *Derandomization of auctions,* Proceedings of the 37th Annual ACM Symposium on Theory of Computing, 619–625, ACM, New York, 2005.

[2] N. Alon, *Asynchronus Threshold Networks,* Graphs and Combinatorics 1 (1985), 305–310.

[3] N. Alon, *Problems and results in extremal combinatorics – II,* Discrete Mathematics 308 (2008), 4460–4472.

[4] S. Aravamuthan and S. Lodha, *Covering codes for Hats-on-a-line,* The Electronic Journal of Combinatorics 13 (2006), Research Paper 21, 12 pp.

[5] J. Aspnes, R. Beigel, M. Furst, and S. Rudich, *The expressive power of voting polynomials,* Combinatorica 14 (1994), 135–148.

[6] R. Beigel, L. Fortnow, and F. Stephan, *Infinitely-often autoreducible sets,* SIAM Journal on Computing 36 (2006), 595–608.

[7] *Berkeley Riddles,* `www.ocf.berkeley.edu/~wwu/riddles/hard.shtml`.

[8] M. Bernstein, *The hat problem and Hamming codes,* MAA Focus, November, 2001, 4–6.

[9] W. Blum, *Denksport für Hutträger,* Die Zeit, May 3, 2001.

[10] J. Blowers, *Hats and hangar queens,* `http://jimvb.home.mindspring.com/hatqueen.htm`.

[11] M. Breit, D. Deshommes, and A. Falden, *Hats required: perfect and imperfect strategies for the hat problem,* manuscript.

[12] E. Brown, K. Mellinger, *Kirkman's schoolgirls wearing hats and walking through fields of numbers,* Mathematics Magazine 82 (2009), no. 1, 3–15.

[13] J. Buhler, *Hat tricks,* Mathematical Intelligencer 24 (2002), no. 4, 44–49.

[14] E. Burke, S. Gustafson, and G. Kendall, *A Puzzle to challenge genetic programming,* Genetic Programming, 136–147, Lecture Notes in Computer Science, Springer, 2002.

[15] S. Butler, M. Hajianghayi, R. Kleinberg, and T. Leighton, *Hat guessing games,* SIAM Journal on Discrete Mathematics 22 (2008), 592–605.

[16] L. Calloway, J. Casher, and S. Hoehn, *From guessing games to compact discs: some applications of coding theory,* manuscript.

[17] G. Cohen, I. Honkala, S. Litsyn, and A. Lobstein, *Covering Codes,* North Holland, 1997.

[18] J. Demongeot, E. Goles, M. Tchuente, *Dynamical Systems and Cellular Automata,* Academic Press, London 1985.

[19] N. Do, *Communicating with eyes, hats and light bulbs,* The Australian Mathematical Society – Gazette 33 (2006), no. 3, 157–164.

[20] T. Ebert, *Applications of recursive operators to randomness and complexity,* Ph.D. Thesis, University of California at Santa Barbara, 1998.

[21] T. Ebert and W. Merkle, *Autoreducibility of random sets: a sharp bound on the density of guessed bits,* Mathematical foundations of computer science 2002, 221–233, Lecture Notes in Computer Science, 2420, Springer, Berlin, 2002.

[22] T. Ebert, W. Merkle, and H. Vollmer, *On the autoreducibility of random sequences,* SIAM Journal on Computing 32 (2003), 1542–1569.

[23] T. Ebert and H. Vollmer, *On the autoreducibility of random sequences,* Mathematical foundations of computer science 2000 (Bratislava), 333–342, Lecture Notes in Computer Science, 1893, Springer, Berlin, 2000.

[24] U. Feige, *On optimal strategies for a hat game on graphs,* submitted.

[25] U. Feige, *You can leave your hat on (if you guess its color),* Technical Report MCS04-03, Computer Science and Applied Mathematics, The Weizmann Institute of Science, 2004, 10 pp.

[26] J. French, *A formal theory of social power,* Psychological Review 63 (1956), 181–194.

[27] A. Frieze and D. Sleator, *Puzzle 15: hat problems,* Alan and Danny's puzzle page, `www.cs.cmu.edu/puzzle`.

[28] D. Gale and I. Adler, *Colored gaps and covering codes,* manuscript, 2001.

[29] M. Gardner, *The Second Scientific American Book of Mathematical Puzzles and Diversions,* University of Chicago Press, Chicago, 1987.

[30] W. Guo, S. Kasala, M. Rao, and B. Tucker, *The hat problem and some variations,* Advances in distribution theory, order statistics, and inference, 459–479, Statistics for Industry and Technology, Birkhäuser Boston, 2007.

[31] F. Harary, *A criterion for unanimity in French's theory of social power,* Studies in Social Power, 168–182, University of Michigan Press, Ann Arbor, 1959.

[32] C. Hardin and A. Taylor, *An introduction to infinite hat problems,* Mathematical Intelligencer 30 (2008), no. 4, 20–25.

[33] C. Hardin and A. Taylor, *Hat problem observations,* manuscript.

[34] A. Hoffman and J. Kruskal, *Integral boundary points of convex polyhedra,* Linear inequalities and related systems, 223–246, Annals of Mathematics Studies, no. 38, Princeton University Press, Princeton, 1956.

[35] N. Immorlica, *Computing with strategic agents,* Ph.D. Thesis, Massachusetts Institute of Technology, 2005.

[36] M. Krzywkowski, *A modified hat problem,* submitted.

[37] M. Krzywkowski, *A more colorful hat problem,* submitted.

[38] M. Krzywkowski, *Hat problem on a graph,* Mathematica Pannonica 21 (2010), 3–21.

[39] M. Krzywkowski, *Hat problem on the cycle C_4,* International Mathematical Forum 5 (2010), 205–212.

[40] H. Lenstra and G. Seroussi, *On hats and other covers,* IEEE International Symposium on Information Theory, Lausanne, 2002.

[41] S. Litsyn, *Simon Litsyn's online table of covering codes,* `www.eng.tau.ac.il/~litsyn/tablecr`.

[42] J. O'Connor and E. Robertson, *Hamming biography,* `www-history.mcs.st-and.ac.uk/Biographies/Hamming.html`.

[43] S. Poljak and M. Sura, *On periodical behavior in societies with symmetric influences,* Combinatorica 3 (1983), 119–121.

[44] J. Poulos, *Could you solve this $1 million hat trick?,* abcNews, November 29, 2001.

[45] F. Roberts, *Discrete Mathematical Models, with Application to Social, Biological, and Environmental Problems,* Prentice-Hall, Englewood Cliffs, 1976.

[46] S. Robinson, *Why mathematicians now care about their hat color,* The New York Times, Science Times Section, page D5, April 10, 2001.

[47] S. Roman, *Coding and Information Theory,* Springer, New York, 1992.

[48] D. Sasha, *Crowns of the Minotaur,* Scientific American, October 2001, p. 82.

[49] P. Winkler, *Games people don't play,* Puzzlers' Tribute, 301–313, A.K. Peters, Natick, Massachusetts, 2001.

[50] P. Winkler, *Mathematical Puzzles: A Connoisseur's Collection,* A.K. Peters, Wellesley, Massachusetts, 2004.

[51] P. Winkler, *Mathematical Mind-Benders,* A.K. Peters, Wellesley, Massachusetts, 2007.