A more colorful hat problem

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Abstract

The topic is the hat problem in which each of n players is randomly fitted with a blue or red hat. Then everybody can try to guess simultaneously his own hat color by looking at the hat colors of the other players. The team wins if at least one player guesses his hat color correctly, and no one guesses his hat color wrong; otherwise the team loses. The aim is to maximize the probability of winning. We consider a generalized hat problem with $q \geq 2$ colors. We solve the problem with three players and three colors. Next we prove some upper bounds on the chance of success of any strategy for the generalized hat problem with n players and q colors. We also consider the numbers of strategies that suffice to be examined to solve the hat problem, or the generalized hat problem.

Keywords: hat problem.
$\mathcal{AMS}$ **Subject Classification:** 91A12.

1 Introduction

In the hat problem, a team of n players enters a room and a blue or red hat is randomly placed on the head of each player. Each player can see the hats of all of the other players but not his own. No communication of any sort is allowed, except for an initial strategy session before the game begins. Once they have had a chance to look at the other hats, each player must simultaneously guess the color of his own hat or pass. The team wins if at least one player guesses his hat color correctly and no one guesses his hat color wrong; otherwise the team loses. The aim is to maximize the probability of winning.

The hat problem with seven players, called the “seven prisoners puzzle”, was formulated by T. Ebert in his Ph.D. Thesis [13]. The hat problem was also the subject of articles in The New York Times [22], Die Zeit [7], and abcNews [21]. It is also one of the Berkeley Riddles [5].

The hat problem with $2^k - 1$ players was solved in [15], and for 2^k players in [12]. The problem with n players was investigated in [8]. The hat problem and Hamming codes were the subject of [9]. The generalized hat problem with n people and q colors was investigated in [20].

There are known many variations of the hat problem. For example in the papers [1, 11, 19] there was considered a variation in which passing is not allowed, thus everybody has to guess his hat color. The aim is to maximize the number of correct guesses. The authors of [17] investigated several variations of the hat problem in which the aim is to design a strategy guaranteeing a desired number of correct guesses. In [18] there was considered a variation in which the probabilities of getting hats of each colors do not have to be equal. The authors of [3] investigated a problem similar to the hat problem, in that paper there are n players which have random bits on foreheads, and they have to vote on the parity of the n bits.

The hat problem and its variations have many applications and connections to different areas of science, for example: information technology [6], linear programming [17], genetic programming [10], economics [1, 19], biology [18], approximating Boolean functions [3], and autoreducibility of random sequences [4, 13–16].

In this paper we consider a generalized hat problem with $q \geq 2$ colors which was first investigated in [20]. Every player has got a hat of one from q possible colors, and the probabilities of getting hats of all colors are equal. We solve the problem with three players and three colors. Next we prove some upper bounds on the chance of success of any strategy for the generalized hat problem with n players and q colors. We also consider the numbers of strategies that suffice to be examined to solve the hat problem, or the generalized hat problem.

2 Preliminaries

First, let us observe that we can confine to deterministic strategies (that is, strategies such that the decision of each player is determined uniquely by the hat colors of the other players). We can do this since for any randomized (not deterministic) strategy there exists a not worse deterministic one. It is true, because every randomized strategy is a convex combination of some deterministic strategies. The probability of winning is a linear function on the convex polyhedron corresponding to the set of all randomized strategies which can be achieved combining those deterministic strategies. It is well known that this function achieves its maximum on a vertex of the polyhedron which corresponds to a deterministic strategy.

Let $\{v_1, v_2, \ldots, v_n\}$ mean a set of players. By $Sc = \{1, 2, \ldots, q\}$ we denote the set of colors.

By a case for the hat problem with n players and q colors we mean a function $c\colon \{v_1, v_2, \ldots, v_n\} \to \{1, 2, \ldots, q\}$, where $c(v_i)$ means the hat color of player v_i. The set of all cases for the hat problem with n players and q colors we denote by $C(n, q)$, of course $|C(n, q)| = q^n$. If $c \in C(n, q)$, then to simplify notation, we write $c = c(v_1)c(v_2)\ldots c(v_n)$ instead of $c = \{(v_1, c(v_1)), (v_2, c(v_2)), \ldots, (v_n, c(v_n))\}$. For example, if a case $c \in C(4, 3)$ is such that $c(v_1) = 2$, $c(v_2) = 3$, $c(v_3) = 1$, and $c(v_4) = 2$, then we write $c = 2312$.

By a situation of a player v_i we mean a function $s_i\colon \{v_1, v_2, \ldots, v_n\} \to Sc \cup \{0\}$, where $s_i(v_j) \in Sc$ if $i \neq j$, while $s_i(v_i) = 0$. The set of all possible situations of v_i in the hat problem with n players and q colors we denote by $St_i(n, q)$, of course $|St_i(n, q)| = q^{n-1}$. If $s_i \in St_i(n, q)$, then for simplicity of notation, we write $s_i = s_i(v_1)s_i(v_2)\ldots s_i(v_n)$ instead of $s_i = \{(v_1, s_i(v_1)), (v_2, s_i(v_2)), \ldots, (v_n, s_i(v_n))\}$. For example, if $s_2 \in St_2(4, 3)$ is such that $s_2(v_1) = 3$, $s_2(v_3) = 4$, and $s_2(v_4) = 2$, then we write $s_2 = 3042$.

We say that a case c corresponds to a situation s_i of player v_i if $c(v_j) = s_i(v_j)$, for every $j \neq i$. This implies that a case corresponds to a situation of v_i if every player excluding v_i in the case has a hat of the same color as in the situation. Of course, to every situation correspond exactly q cases.

By a guessing instruction of a player v_i we mean a function $g_i\colon St_i(n, q) \to Sc \cup \{*\}$, which for a given situation gives the color v_i guesses his hat is if $g_i(s_i) \neq *$, otherwise v_i passes. Thus a guessing instruction is a rule determining the behavior of a player in every situation.

Let c be a case, and let s_i be the situation (of player v_i) corresponding to this case. The guess of v_i in the case c is correct (wrong, respectively) if $g_i(s_i) = c(v_i)$ ($* \neq g_i(s_i) \neq c(v_i)$, respectively). By result of the case c we mean a win if at least one player guesses his hat color correctly, and no player guesses his hat color wrong, that is, $g_i(s_i) = c(v_i)$ (for some i) and there is no j such that $* \neq g_j(s_j) \neq c(v_j)$. Otherwise the result of the case c is a loss.

By a strategy we mean a sequence $(g_1, g_2, \ldots, g_n)$, where g_i is the guessing instruction of player v_i. The family of all strategies for the hat problem with n players and q colors we denote by $\mathcal{F}(n, q)$.

If $S \in \mathcal{F}(n, q)$, then the set of cases for which the team wins using the strategy S we denote by $W(S)$. Consequently, by the chance of success of the strategy S we mean the number $p(S) = |W(S)|/|C(n, q)|$. We define $h(n, q) = \max\{p(S) \colon S \in \mathcal{F}(n, q)\}$. We say that a strategy S is optimal for the hat problem with n players and q colors if $p(S) = h(n, q)$.

By solving the hat problem with n players and q colors we mean finding the number $h(n, q)$.

3 Hat problem with three players and three colors

In this section we solve the hat problem with three players and three colors.

We say that a strategy is symmetric if every player makes his decision on the basis of only numbers of hats of each color seen by him, and all players behave in the same way. A strategy is nonsymmetric if it is not symmetric.

The authors of [18] solved the hat problem with three players and three colors by giving a symmetric strategy found by computer, and proving that it is optimal.

We solve this problem by proving the optimality of a nonsymmetric strategy found without using computer.

Let us consider the following strategy for the hat problem with three players and three colors.

Strategy 1 *Let $S = (g_1, g_2, g_3) \in \mathcal{F}(3, 3)$ be the strategy as follows.*

$$g_1(s_1) = \begin{cases} s_1(v_3) & \text{if } s_1(v_2) \neq s_1(v_3), \\ * & \text{otherwise;} \end{cases}$$

$$g_2(s_2) = \begin{cases} s_2(v_3) & \text{if } s_2(v_1) \neq s_2(v_3), \\ * & \text{otherwise;} \end{cases}$$

$$g_3(s_3) = \begin{cases} s_3(v_1) & \textit{if } s_3(v_1) = s_3(v_2), \\ * & \textit{otherwise.} \end{cases}$$

It means that players proceed as follows.

- **The player** v_1. If v_2 and v_3 have hats of different colors, then he guesses he has a hat of the color v_3 has, otherwise he passes.

- **The player** v_2. If v_1 and v_3 have hats of different colors, then he guesses he has a hat of the color v_3 has, otherwise he passes.

- **The player** v_3. If v_1 and v_2 have hats of the same color, then he guesses he has a hat of the color they have, otherwise he passes.

All cases we present in Table 1, where the symbol $+$ means correct guess (success), $-$ means wrong guess (loss), and blank square means passing.

For example, in the first case the player v_1 sees two hats of the same color, so he passes. By the same reason the player v_2 also passes. The player v_3 sees two hats of the first color, so he guesses he has a hat of the first color. Since v_3 has a hat of the first color, the guess is correct, and the result of the case is a win.

In the second case the player v_1 sees two hats of different colors, so he guesses he has a hat of the color v_3 has. Since v_1 and v_3 have hats of different colors, the guess is wrong, and the result of the case is a loss. Additionally, the player v_2 guesses his hat color wrong by the same reason as v_1. Moreover, the guess of v_3 is also wrong. The player v_3 sees two hats of the first color, so he guesses he has a hat of the first color. The guess is wrong, as v_3 has a hat of the second color.

In the fourth case the player v_1 sees two hats of different colors, so he guesses he has a hat of the color v_3 has. Since v_1 and v_3 have hats of the same color, the guess is correct. The player v_2 sees two hats of the same color, so he passes. The player v_3 sees two hats of different colors, so he passes. This implies that the result of the case is a win.

In the sixth case the player v_1 sees two hats of different colors, so he guesses he has a hat of the color v_3 has. Since v_1 and v_3 have hats of different colors, the guess is wrong, and the result of the case is a loss. Additionally, the player v_2 guesses his hat color wrong by reasons similar as v_1. The player v_3 passes, as he sees two hats of different colors.

No	The color of the hat of			The guess of			Result
	v_1	v_2	v_3	v_1	v_2	v_3	
1	1	1	1			$+$	$+$
2	1	1	2	$-$	$-$	$-$	$-$
3	1	1	3	$-$	$-$	$-$	$-$
4	1	2	1	$+$			$+$
5	1	2	2		$+$		$+$
6	1	2	3	$-$	$-$		$-$
7	1	3	1	$+$			$+$
8	1	3	2	$-$	$-$		$-$
9	1	3	3		$+$		$+$
10	2	1	1		$+$		$+$
11	2	1	2	$+$			$+$
12	2	1	3	$-$	$-$		$-$
13	2	2	1	$-$	$-$	$-$	$-$
14	2	2	2			$+$	$+$
15	2	2	3	$-$	$-$	$-$	$-$
16	2	3	1	$-$	$-$		$-$
17	2	3	2	$+$			$+$
18	2	3	3		$+$		$+$
19	3	1	1		$+$		$+$
20	3	1	2	$-$	$-$		$-$
21	3	1	3	$+$			$+$
22	3	2	1	$-$	$-$		$-$
23	3	2	2		$+$		$+$
24	3	2	3	$+$			$+$
25	3	3	1	$-$	$-$	$-$	$-$
26	3	3	2	$-$	$-$	$-$	$-$
27	3	3	3			$+$	$+$

Table 1

Counting the plusses in the last column, we get the following observation.

Observation 2 *Using Strategy 1 the team wins for* 15 *of* 27 *cases.*

Now, we solve the hat problem with three players and three colors.

Fact 3 $h(3,3) = 5/9$.

Proof. Since using Strategy 1 the team wins for 15 of 27 cases, we have $h(3,3) \geq 15/27 = 5/9$. Suppose that $h(3,3) > 5/9$, that is, there exists a strategy such that the team wins for more than 15 cases. Let S be any strategy for the hat problem with three players and three colors. Any guess made by any player in any situation is wrong in exactly two cases, because to any situation of any player correspond three cases, and in exactly two of them this player has a hat of a color different than the one he guesses. In the strategy S every player guesses his hat color in at most 5 situations, because if some player guesses his hat color in at least 6 situations, then the team loses for at least 12 cases, and wins for at most 15 cases, a contradiction. Any guess made by any player in any situation is correct in exactly one case, because to any situation of any player correspond three cases, and in exactly one of them this player has a hat of the color he guesses. There are three players, every one of them guesses his hat color in at most five cases, and every guess is correct in exactly one case. Therefore using the strategy S the team wins for at most 15 cases, a contradiction. ■

4 Hat problem with n players and q colors

Now we consider the generalized hat problem with n players and q colors. Noga Alon [2] has proven that for this problem there exists a strategy such that the chance of success is greater than or equal to

$$1 - \frac{1 + (q-1)\log n}{n} - \left(1 - \frac{1}{q}\right)^n .$$

First we prove an upper bound on the number of cases for which the team wins using any strategy for the problem.

Theorem 4 *If S is a strategy for the hat problem with n players and q colors, then*

$$|W(S)| \leq n \left\lfloor \frac{q^n - |W(S)|}{q-1} \right\rfloor .$$

Proof. Any guess made by any player in any situation is wrong in exactly $q-1$ cases, because to any situation of any player correspond q cases, and in exactly $q-1$ of them this player has a hat of a color different than the one he guesses.

Let us consider any player. The number of situations in which he guesses his hat color in the strategy S cannot be neither greater than nor equal to

$$\left\lfloor \frac{q^n - |W(S)|}{q-1} \right\rfloor + 1,$$

otherwise the number of cases in which he guesses his hat color wrong is greater than or equal to

$$(q-1)\left(\left\lfloor \frac{q^n - |W(S)|}{q-1} \right\rfloor + 1\right).$$

It is more than

$$(q-1)\left(\frac{q^n - |W(S)|}{q-1}\right) = q^n - |W(S)|.$$

This implies that the team loses for more than $q^n - |W(S)|$ cases, and therefore the number of cases for which the team wins is less than

$$|C(n,q)| - (q^n - |W(S)|) = q^n - q^n + |W(S)| = |W(S)|.$$

This is a contradiction, as $|W(S)|$ is the number of cases for which the team wins. Any guess made by any player in any situation is correct in exactly one case, because to any situation of any player correspond q cases, and in exactly one of them this player has a hat of the color he guesses. This implies that the number of cases for which the team wins using the strategy S is at most

$$n \left\lfloor \frac{q^n - |W(S)|}{q-1} \right\rfloor.$$

■

Now we give an equivalent upper bound on the chance of success of any strategy for the hat problem with n players and q colors, which is easy to prove.

Theorem 5 *Let S be any strategy for the hat problem with n players and q colors. Then*

$$p(S) \le \frac{n}{q^n} \left\lfloor \frac{q^n - q^n \cdot p(S)}{q-1} \right\rfloor.$$

Now we see that Fact 3 follows from Theorem 4, as well as from Theorem 5. We show that it follows from Theorem 4.

Proof (of Fact 3). Since using Strategy 1 the team wins for 15 of 27 cases, by definition we get $h(3,3) \ge p(S) = 15/27 = 5/9$. Now we prove that $h(3,3) \le 5/9$.

Let S be an optimal strategy for the hat problem with three players and three colors. By Theorem 4 we have

$$|W(S)| \leq 3 \left\lfloor \frac{27 - |W(S)|}{2} \right\rfloor.$$

This implies that

$$|W(S)| \leq 3 \cdot \frac{27 - |W(S)|}{2} = 40.5 - 3|W(S)|/2.$$

Now we easily get $|W(S)| \leq 81/5 = 16.2$. Since $|W(S)|$ is an integer, we have $|W(S)| \leq 16$. If $|W(S)| = 16$, then $16 \leq 3\lfloor(27 - 16)/2\rfloor = 3 \cdot 5 = 15$, a contradiction. This implies that $|W(S)| \leq 15$. Since $|C(3,3)| = 27$, we get $p(S) \leq 15/27 = 5/9$. Since S is an optimal strategy for the hat problem with three players and three colors, by definition we get $h(3,3) = p(S) \leq 5/9$. ■

The next result proven in [20, Proposition 3] is a corollary from Theorem 4 or 5.

Corollary 6 ([20, Proposition 3]) *If S is a strategy for the hat problem with n players and q colors, then*

$$p(S) \leq \frac{n}{n+q-1}.$$

Proof. By Theorem 4 we have

$$|W(S)| \leq n \left\lfloor \frac{q^n - |W(S)|}{q-1} \right\rfloor.$$

This implies that

$$|W(S)| \leq n \cdot \frac{q^n - |W(S)|}{q-1} = \frac{nq^n}{q-1} - |W(S)| \left(\frac{n}{q-1}\right).$$

Consequently,

$$\begin{aligned} |W(S)| \left(1 + \frac{n}{q-1}\right) \leq \frac{nq^n}{q-1} \Leftrightarrow |W(S)| &\leq \frac{q-1}{n+q-1} \cdot \frac{nq^n}{q-1} \\ \Leftrightarrow p(S) = \frac{|W(S)|}{q^n} &\leq \frac{n}{n+q-1}. \end{aligned}$$

■

Now we show that the previous corollary is weaker than Theorem 4, that is, Theorem 4 does not follow from Corollary 6. Let S be any strategy for the hat problem with three players and three colors. By Theorem 4 we have $|W(S)| \leq 15$ (it is shown in the proof of Fact 3 using Theorem 4). Thus

$$p(S) = \frac{|W(S)|}{|C(3,3)|} \leq \frac{15}{3^3} = \frac{5}{9}.$$

By Corollary 6 we get

$$p(S) \leq \frac{n}{n+q-1} = \frac{3}{5}.$$

Since $3/5 > 5/9$, Corollary 6 is weaker than Theorem 4.

Now let us consider the hat problem with two colors ($q = 2$), and any strategy S for this problem. By Corollary 6 we get the upper bound

$$p(S) \leq \frac{n}{n+1}$$

previously given in [15], which is sharp for $n = 2^k - 1$, where k is a positive integer.

5 Number of strategies that suffice to be examined

In this section we consider the number of strategies the examination of which suffices to solve the hat problem, and the generalized hat problem with q colors.

First, we count all possible strategies for the hat problem. We have n players, there are 2^{n-1} possible situations of each one of them, and in each situation there are three possibilities of behavior (to guess the first color, to guess the second color, or to pass). This implies that the number of possible strategies is equal to

$$\left(3^{2^{n-1}}\right)^n.$$

Now we prove that it is not necessary to examine every strategy to solve the problem.

Fact 7 *To solve the hat problem with n players, it suffices to examine*

$$\left(3^{2^{n-1}-2}\right)^n = \left(3^{2^{n-1}}\right)^n \cdot \frac{1}{9^n}$$

strategies.

Proof. Let S be an optimal strategy for the hat problem with n players. If in this strategy no player guesses his hat color, then obviously $p(S) = 0$. This is a contradiction to the optimality of S. Thus in the strategy S some player guesses his hat color. Without loss of generality we assume that this player is v_1, and he guesses his hat color in the situation $011\dots1$. Additionally, without loss of generality we assume that in this situation he guesses he has a hat of the second color. This guess is wrong in the case $11\dots1$, causing the loss of the team. Thus the result of this case cannot be made worse. If some player other than v_1, say v_i, guesses he has the second color when he sees only hats of the first color, then his guess is wrong in the case $11\dots1$, and is correct in the case when v_i has the second color and all the remaining vertices have the first color. Since it cannot make worse the chance of success, we may assume that every player excluding v_i guesses he has a hat of the second color when he sees hats only of the first color. Assume that some player, say v_i, guesses his hat color when he sees one hat of the second color and $n-2$ hats of the first color. If in this situation he guesses he has a hat of the first color, then in the case corresponding to that situation, and in which he has a hat of the first color, his guess is correct, as well as the guess of the player who has a hat of the second color. Since it cannot improve the chance of success, we may assume that in this situation v_i does not guess he has a hat of the first color. If in that situation he guesses he has a hat of the second color, then in the case corresponding to that situation, and in which he has a hat of the first color, his guess is wrong, while at the same time the guess of the player who has a hat of the second color is correct. Since it makes the guess of this player pointless, we may assume that in that situation v_i does not guess he has a hat of the second color. This implies that we may assume that every player who sees one hat of the second color and $n-2$ hats of the first color, passes. Now we conclude that for each player we can assume his behavior in two situations. This implies that for each player there are two situations less to consider. In this way we get the desired number. ■

Now, we count all possible strategies for the generalized hat problem with q colors. We have n players, there are q^{n-1} possible situations of each one of them, and in each situation there are $q+1$ possibilities of behavior (to guess one of the q colors, or to pass). This implies that the number of possible strategies is equal to

$$\left((q+1)^{q^{n-1}}\right)^n.$$

Now we prove that it is not necessary to examine every strategy to solve the problem.

Fact 8 *To solve the hat problem with n players and q colors, it suffices to examine*

$$\left((q+1)^{q^{n-1}-1}\right)^n = \left((q+1)^{q^{n-1}}\right)^n \cdot \frac{1}{(q+1)^n}$$

strategies.

Proof. Let S be an optimal strategy for the hat problem with n players and q colors. If in this strategy no player guesses his hat color, then obviously $p(S) = 0$. This is a contradiction to the optimality of S. Thus in the strategy S some player guesses his hat color. Without loss of generality we assume that this player is v_1, and he guesses his hat color in the situation $011\ldots1$. Additionally, without loss of generality we assume that in this situation he guesses he has a hat of the second color. Let v_i be any player other than v_1. If in this situation v_i guesses he has a hat of the first color, then in the case corresponding to that situation, and in which he has a hat of the first color, his guess is correct, as well as the guess of v_1. Since it cannot improve the chance of success, we may assume that in this situation v_i does not guess he has a hat of the first color. If in that situation v_i guesses he has a hat of any color other than the first, then in the case corresponding to that situation, and in which he has a hat of the first color, his guess is wrong, while at the same time the guess of v_1 is correct. Since it makes the guess of v_1 pointless, we may assume that in that situation v_i does not guess any color other that the first. This implies that we may assume that every player other than v_1 in the situation in which v_1 has a hat of the second color, and all the remaining players have hats of the first color, passes. Now we conclude that for each player we can assume his behavior in one situation. This implies that for each player there is one situation less to consider. In this way we get the desired number. ∎

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