A lower bound on the total outer-independent domination number of a tree

Une borne inférieure pour le cardinal des sous-ensembles totalement dominants et extérieurement-indépendants d'un arbre

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Abstract

A total outer-independent dominating set of a graph G is a set D of vertices of G such that every vertex of G has a neighbor in D, and the set $V(G) \setminus D$ is independent. The total outer-independent domination number of a graph G, denoted by $\gamma_t^{oi}(G)$, is the minimum cardinality of a total outer-independent dominating set of G. We prove that for every nontrivial tree T of order n with l leaves we have $\gamma_t^{oi}(T) \geq (2n - 2l + 2)/3$, and we characterize the trees attaining this lower bound.

Résumé

Un sous-ensemble totalement dominant et extérieurement indépendant d'un graphe est un sous-ensemble D des sommets de G tel que chaque sommet de G ait un voisin dans D et l'ensemble $V(G) \setminus D$ soit indépendant. Le plus petit cardinal d'un tel sous-ensemble est noté $\gamma_t^{oi}(G)$. Nous démontrons que pour tout arbre T non trivial, d'ordre n avec l feuilles, nous avons $\gamma_t^{oi}(T) \geq (2n - 2l + 2)/3$. De plus, nous caractérisons les arbres réalisant cette borne inférieure.

1 Introduction

Let $G = (V, E)$ be a graph. By the neighborhood of a vertex v of G we mean the set $N_G(v) = \{u \in V(G) \colon uv \in E(G)\}$. The degree of a vertex v, denoted by $d_G(v)$, is the cardinality of its neighborhood. By a leaf we mean a vertex of degree one, while a support vertex is a vertex adjacent to a leaf. The path on n vertices we denote by P_n. We say that a subset of $V(G)$ is independent if there is no edge between every two its vertices. By a star we mean a connected graph in which exactly one vertex has degree greater than one. By a double star we mean a graph obtained from a star by joining a positive number of vertices to one of its leaves.

A subset $D \subseteq V(G)$ is a dominating set of G if every vertex of $V(G) \setminus D$ has a neighbor in D, while it is a total dominating set of G if every vertex of G has a neighbor in D. The domination (total domination, respectively) number of G, denoted by $\gamma(G)$ ($\gamma_t(G)$, respectively), is the minimum cardinality of a dominating (total dominating, respectively) set of G. Total domination in graphs was introduced by Cockayne, Dawes, and Hedetniemi [2]. For a comprehensive survey of domination in graphs, see [3, 4].

A subset $D \subseteq V(G)$ is a total outer-independent dominating set, abbreviated TOIDS, of G if every vertex of G has a neighbor in D, and the set $V(G) \setminus D$ is independent. The total outer-independent domination number of G, denoted by $\gamma_t^{oi}(G)$, is the minimum cardinality of a total outer-independent dominating set of G. A total outer-independent dominating set of G of minimum cardinality is called a $\gamma_t^{oi}(G)$-set. The study of total outer-independent domination in graphs was initiated in [5].

Chellali and Haynes [1] established the following lower bound on the total domination number of a tree. For every nontrivial tree T of order n with l leaves we have $\gamma_t(T) \geq (n - l + 2)/2$. They also characterized the extremal trees.

We prove the following lower bound on the total outer-independent domination number of a tree. For every nontrivial tree T of order n with l leaves we have $\gamma_t^{oi}(T) \geq (2n - 2l + 2)/3$. We also characterize the trees attaining this lower bound.

2 Results

We begin with the following two straightforward observations.

Observation 1 *Every support vertex of a graph G is in every $\gamma_t^{oi}(G)$-set.*

Observation 2 *For every connected graph G of diameter at least three there exists a $\gamma_t^{oi}(G)$-set that contains no leaf.*

We show that if T is a nontrivial tree of order n with l leaves, then $\gamma_t^{oi}(T)$ is bounded below by $(2n - 2l + 2)/3$. For the purpose of characterizing the trees attaining this bound we introduce a family $\mathcal{T}$ of trees $T = T_k$ that can be obtained as follows. Let T_1 be a path P_4 with support vertices labeled x and y, and let $A(T_1) = \{x, y\}$. Let H be a path P_3 with a leaf labeled u, and the support vertex labeled v. If k is a positive integer, then T_{k+1} can be obtained recursively from T_k by one of the following operations.

- Operation $\mathcal{O}_1$: Attach a vertex by joining it to any vertex of $A(T_k)$. Let $A(T_{k+1}) = A(T_k)$.
- Operation $\mathcal{O}_2$: Attach a copy of H by joining u to any leaf of T_k. Let $A(T_{k+1}) = A(T_k) \cup \{u, v\}$.

Now we prove that for every tree T of the family $\mathcal{T}$, the set $A(T)$ defined above is a TOIDS of minimum cardinality equal to $(2n - 2l + 2)/3$.

Lemma 3 *If $T \in \mathcal{T}$, then the set $A(T)$ defined above is a $\gamma_t^{oi}(T)$-set of size $(2n - 2l + 2)/3$.*

Proof. We use the terminology of the construction of the trees $T = T_k$, the set $A(T)$, and the graph H defined above. To show that $A(T)$ is a $\gamma_t^{oi}(T)$-set of cardinality $(2n - 2l + 2)/3$ we use the induction on the number k of operations performed to construct T. If $T = T_1 = P_4$, then $(2n - 2l + 2)/3 = (8 - 4 + 2)/3 = 2 = |A(T)| = \gamma_t^{oi}(T)$. Let $k \geq 2$ be an integer. Assume that the result is true for every tree $T' = T_k$ of the family $\mathcal{T}$ constructed by $k - 1$ operations. Let n' mean the order of the tree T' and l' the number of its leaves. Let $T = T_{k+1}$ be a tree of the family $\mathcal{T}$ constructed by k operations.

If T is obtained from T' by operation $\mathcal{O}_1$, then $n = n' + 1$. Observe that $A(T')$ contains no leaf. Thus $l = l' + 1$. It is easy to see that $A(T) = A(T')$ is a TOIDS of the tree T. Thus $\gamma_t^{oi}(T) \leq |A(T)| = |A(T')| = \gamma_t^{oi}(T')$. Of course, $\gamma_t^{oi}(T) \geq \gamma_t^{oi}(T')$. This implies that $\gamma_t^{oi}(T) = |A(T)| = |A(T')| = (2n' - 2l' + 2)/3 = (2n - 2 - 2l + 2 + 2)/3 = (2n - 2l + 2)/3$.

Now assume that T is obtained from T' by operation $\mathcal{O}_2$. We have $n = n' + 3$ and $l = l'$. It is easy to see that $A(T) = A(T') \cup \{u, v\}$ is a TOIDS of the tree T. Thus $\gamma_t^{oi}(T) \le |A(T)| = |A(T')| + 2 = \gamma_t^{oi}(T') + 2$. The neighbor of u another than v let us denote by w, and its neighbor another than u let us denote by x. First assume that there exists a $\gamma_t^{oi}(T)$-set that does not contain w. Thus $u, v \in D$. It is easy to see that $D \setminus \{u, v\}$ is a TOIDS of the tree T'. Now assume that every $\gamma_t^{oi}(T)$-set contains w. Since $\mathrm{diam}(T) \ge 3$, let D be a $\gamma_t^{oi}(T)$-set that contains no leaf. Thus $u, v \in D$. If $x \in D$, then it is easy to see that $D \setminus \{u, v\}$ is a TOIDS of the tree T'. Now suppose that $x \notin D$. Since $T' \in \mathcal{T}$, we have $T' \ne P_2$. This implies that $d_{T'}(x) = d_T(x) \ge 2$. Since $x \notin D$ and the set $V(T) \setminus D$ is independent, every neighbor of x belongs to the set D. Let us observe that $D \cup \{x\} \setminus \{w\}$ is a TOIDS of the tree T that does not contain w, a contradiction. Since in every case $D \setminus \{u, v\}$ is a TOIDS of the tree T', we get $\gamma_t^{oi}(T') \le \gamma_t^{oi}(T) - 2$. Now we conclude that $\gamma_t^{oi}(T) = \gamma_t^{oi}(T') + 2$. We get $\gamma_t^{oi}(T) = |A(T)| = \gamma_t^{oi}(T') + 2 = |A(T') \cup \{u, v\}| = (2n' - 2l' + 2)/3 + 2 = (2n - 6 - 2l + 2 + 6)/3 = (2n - 2l + 2)/3$. ■

Now we establish the main result, a lower bound on the total outer-independent domination number of a tree together with the characterization of the extremal trees.

Theorem 4 *If T is a nontrivial tree of order n with l leaves, then $\gamma_t^{oi}(T) \ge (2n - 2l + 2)/3$ with equality if and only if $T \in \mathcal{T}$.*

Proof. If $\mathrm{diam}(T) = 1$, then $T = P_2$. We have $(2n - 2l + 2)/3 = (4 - 4 + 2)/3 = 2/3 < 2 = \gamma_t^{oi}(T)$. If $\mathrm{diam}(T) = 2$, then T is a star $K_{1,m}$. We have $n = m + 1$ and $l = m$. Now we get $(2n - 2l + 2)/3 = (2m + 2 - 2m + 2)/3 = 4/3 < 2 = \gamma_t^{oi}(T)$. Now let us assume that $\mathrm{diam}(T) = 3$. Thus T is a double star. If $T = P_4$, then $T \in \mathcal{T}$, and by Lemma 3 we have $\gamma_t^{oi}(T) = (2n - 2l + 2)/3$. Now assume that T is a double star different than P_4. By Observation 1, for any double star T^* of the family $\mathcal{T}$ both support vertices belong to every $\gamma_t^{oi}(T^*)$-set. Lemma 3 implies that they belong to the set $A(T^*)$ defined earlier. Therefore the tree T can be obtained from P_4 by proper numbers of operations $\mathcal{O}_1$ performed on the support vertices. Thus $T \in \mathcal{T}$. By Lemma 3 we have $\gamma_t^{oi}(T) = (2n - 2l + 2)/3$.

Now we assume that $\mathrm{diam}(T) \ge 4$. Thus the order of the tree T is an integer $n \ge 5$. The result we obtain by the induction on the number n. Assume that the theorem is true for every tree T' of order $n' < n$ with l' leaves.

First assume that some support vertex of T, say x, is adjacent to at least two leaves. One of them let us denote by y. Let $T' = T - y$. We have $n' = n - 1$ and $l' = l - 1$. Since every $\gamma_t^{oi}(T')$-set, as well as every $\gamma_t^{oi}(T)$-set, contains every support vertex, it is easy to observe that $\gamma_t^{oi}(T) = \gamma_t^{oi}(T')$. Now we get $\gamma_t^{oi}(T) = \gamma_t^{oi}(T') \ge (2n' - 2l' + 2)/3 = (2n - 2 - 2l + 2 + 2)/3 = (2n - 2l + 2)/3$. If $\gamma_t^{oi}(T) = (2n - 2l + 2)/3$, then obviously $\gamma_t^{oi}(T') = (2n' - 2l' + 2)/3$. By the inductive hypothesis we have $T' \in \mathcal{T}$. By Observation 1, the vertex x is in every TOIDS of the tree T'. Lemma 3 implies that $x \in A(T')$. Therefore the tree T can be obtained from T' by operation $\mathcal{O}_1$. Thus $T \in \mathcal{T}$. Henceforth, we can assume that every support vertex of T is adjacent to exactly one leaf.

We now root T at a vertex r of maximum eccentricity $\mathrm{diam}(T)$. Let v be a support vertex at maximum distance from r, u be the parent of v, and w be the parent of u in the rooted tree. By T_x let us denote the subtree induced by a vertex x and its descendants in the rooted tree T. We distinguish between the following two cases: $d_T(u) \ge 3$ and $d_T(u) = 2$.

Case 1. $d_T(u) \ge 3$. First assume that u has a child $b \ne v$ that is a support vertex. Let $T' = T - T_v$. We have $n' = n - 2$ and $l' = l - 1$. Let D be a $\gamma_t^{oi}(T)$-set that contains no leaf. Thus $u, v, b \in D$. Of course, $D \setminus \{v\}$ is a TOIDS of the tree T'. Therefore $\gamma_t^{oi}(T') \le \gamma_t^{oi}(T) - 1$. Now we get $\gamma_t^{oi}(T) \ge \gamma_t^{oi}(T') + 1 \ge (2n' - 2l' + 2)/3 + 1 = (2n - 4 - 2l + 2 + 2 + 3)/3 = (2n - 2l + 3)/3 > (2n - 2l + 2)/3$.

Now assume that v is the only one support vertex among the descendants of u. Thus u is a parent of a leaf, say x. Let $T' = T - x$. We have $n' = n - 1$ and $l' = l - 1$. Let D be any $\gamma_t^{oi}(T)$-set. We have $u, v \in D$. It is easy to see that D is a TOIDS of the tree T'. Therefore $\gamma_t^{oi}(T') \leq \gamma_t^{oi}(T)$. Now we get $\gamma_t^{oi}(T) \geq \gamma_t^{oi}(T') \geq (2n' - 2l' + 2)/3 = (2n - 2 - 2l + 2 + 2)/3 = (2n - 2l + 2)/3$. If $\gamma_t^{oi}(T) = (2n - 2l + 2)/3$, then obviously $\gamma_t^{oi}(T') = (2n' - 2l' + 2)/3$. By the inductive hypothesis we have $T' \in \mathcal{T}$. It follows from the definition of the family $\mathcal{T}$ that for every tree $T^* \in \mathcal{T}$ the set $A(T^*)$ does not contain any leaf. Lemma 3 implies that $A(T')$ is a TOIDS of the tree T'. Since v has to have a neighbor in $A(T)$, we have $u \in A(T')$. Therefore the tree T can be obtained from T' by operation $\mathcal{O}_1$. Thus $T \in \mathcal{T}$.

Case 2. $d_T(u) = 2$. We consider the following two possibilities: $d_T(w) = 2$ and $d_T(w) \geq 3$. First assume that $d_T(w) = 2$. The parent of w let us denote by x. If $d_T(x) = 1$, then $T = P_5$. We have $(2n - 2l + 2)/3 = (10 - 4 + 2)/3 = 8/3 < 3 = \gamma_t^{oi}(T)$. Now assume that $T \neq P_5$. Thus $d_T(x) \geq 2$. First let us prove that there exists a $\gamma_t^{oi}(T)$-set that does not contain w. Assume that there exists a $\gamma_t^{oi}(T)$-set D that contains w. If $x \notin D$, then every neighbor of x belongs to D as the set $V(T) \setminus D$ is independent. It is easy to see that $D \cup \{x\} \setminus \{w\}$ is a TOIDS of the tree T of cardinality $|D| = \gamma_t^{oi}(T)$. Thus $D \cup \{x\} \setminus \{w\}$ is a $\gamma_t^{oi}(T)$-set that does not contain w. If $x \in D$, then no neighbor of x besides w belongs to the set D, otherwise $D \setminus \{w\}$ is a TOIDS of the tree T of cardinality $\gamma_t^{oi}(T) - 1$, a contradiction. Let y be any neighbor of x besides w. Observe that $D \cup \{y\} \setminus \{w\}$ is a TOIDS of the tree T of cardinality $|D| = \gamma_t^{oi}(T)$. Thus $D \cup \{y\} \setminus \{w\}$ is a $\gamma_t^{oi}(T)$-set that does not contain w. Now we conclude that there exists a $\gamma_t^{oi}(T)$-set that does not contain w. Let D be such a set. Of course, we have $u, v \in D$. Let $T' = T - T_u$. We have $n' = n - 3$ and $l' = l$. Let us observe that $x \in D$ as $w \notin D$ and the set $V(T) \setminus D$ is independent. Thus $D \setminus \{u, v\}$ is a TOIDS of the tree T'. Therefore $\gamma_t^{oi}(T') \leq \gamma_t^{oi}(T) - 2$. Now we get $\gamma_t^{oi}(T) \geq \gamma_t^{oi}(T') + 2$ $\geq (2n' - 2l' + 2) + 2 = (2n - 6 - 2l + 2 + 6)/3 = (2n - 2l + 2)/3$. If $\gamma_t^{oi}(T) = (2n - 2l + 2)/3$, then we easily get $\gamma_t^{oi}(T') = (2n' - 2l' + 2)/3$. By the inductive hypothesis we get $T' \in \mathcal{T}$. The tree T can be obtained from T' by operation $\mathcal{O}_2$. Thus $T \in \mathcal{T}$.

Now assume that $d_T(w) \geq 3$. First assume some descendant of w is a leaf. Let D be a $\gamma_t^{oi}(T)$-set that contains no leaf. Thus $v, u, w \in D$. The descendant of v let us denote by z. Let $T' = T - z$. We have $n' = n - 1$ and $l' = l$. It is easy to see that $D \setminus \{v\}$ is a TOIDS of the tree T'. Therefore $\gamma_t^{oi}(T') \leq \gamma_t^{oi}(T) - 1$. Now we get $\gamma_t^{oi}(T) \geq \gamma_t^{oi}(T') + 1 \geq (2n' - 2l' + 2)/3 + 1 = (2n - 2 - 2l + 2 + 3)/3 = (2n - 2l + 3)/3 > (2n - 2l + 2)/3$.

Now assume that among the descendants of w there is no leaf. Let x mean a descendant of w different than u. Let $T' = T - T_u$. We have $n' = n - 3$ and $l' = l - 1$. Let D be a $\gamma_t^{oi}(T)$-set that contains no leaf. We have $u, v, x \in D$. Observe that $D \setminus \{u, v\}$ is a TOIDS of the tree T'. Therefore $\gamma_t^{oi}(T') \leq \gamma_t^{oi}(T) - 2$. Now we get $\gamma_t^{oi}(T) \geq \gamma_t^{oi}(T') + 2 \geq (2n' - 2l' + 2)/3 + 2 = (2n - 6 - 2l + 2 + 2 + 6)/3 = (2n - 2l + 4)/3 > (2n - 2l + 2)/3$.

■

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