Bounds on the vertex-edge domination number of a tree

Bornes sur le nombre de domination sommet-arête d'un arbre

B. Krishnakumari*, Y. B. Venkatakrishnan*

Marcin Krzywkowski†‡
marcin.krzywkowski@gmail.com

Abstract

A vertex-edge dominating set of a graph G is a set D of vertices of G such that every edge of G is incident with a vertex of D or a vertex adjacent to a vertex of D. The vertex-edge domination number of a graph G, denoted by $\gamma_{ve}(T)$, is the minimum cardinality of a vertex-edge dominating set of G. We prove that for every tree T of order $n \geq 3$ with l leaves and s support vertices we have $(n-l-s+3)/4 \leq \gamma_{ve}(T) \leq n/3$, and we characterize the trees attaining each of the bounds.

Résumé

Un ensemble sommet-arête dominant d'un graphe G est un ensemble D de sommets de G tel que chaque arête de G est incident avec un sommet de D ou un sommet adjacent á un sommet de D. Le nombre de domination sommet-arête d'un graphe G, noté $\gamma_{ve}(T)$, est le cardinal minimum d'un ensemble sommet-arête dominant de G. Nous prouvons que pour chaque arbre T d'ordre $n \geq 3$ avec l feuilles et des sommets s de soutien que nous avons $(n-l-s+3)/4 \leq \gamma_{ve}(T) \leq n/3$, et nous caractérisons les arbres atteignent chacun des limites.

1 Introduction

Let $G = (V, E)$ be a graph. By the neighborhood of a vertex v of G we mean the set $N_G(v) = \{u \in V(G)\colon uv \in E(G)\}$. The degree of a vertex v, denoted by $d_G(v)$, is the cardinality of its neighborhood. By a leaf we mean a vertex of degree one, while a support vertex is a vertex adjacent to a leaf. We say that a support vertex is strong (weak, respectively) if it is adjacent to at least two leaves (exactly one leaf, respectively). The path on n vertices we denote by P_n. Let T be a tree, and let v be a vertex of T. We say that v is adjacent to a path P_n if there is a neighbor of v, say x, such that the subtree resulting from T by removing the edge vx and which contains the vertex x as a leaf, is a path P_n. By a star we mean a connected graph in which exactly one vertex has degree greater than one.

A subset $D \subseteq V(G)$ is a dominating set of G if every vertex of $V(G) \setminus D$ has a neighbor in D. The domination number of G is the minimum cardinality of a dominating set of G. For a comprehensive survey of domination in graphs, see [2, 3].

An edge $e \in E(G)$ is vertex-edge dominated by a vertex $v \in V(G)$ if e is incident to v, or e is adjacent to an edge incident to v. A subset $D \subseteq V(G)$ is a vertex-edge dominating set, abbreviated VEDS, of G if every edge of G is vertex-edge dominated by a vertex of D. The vertex-edge domination number of G, denoted by $\gamma_{ve}(G)$, is the minimum cardinality of a vertex-edge dominating set of G. A vertex-edge dominating set of G of minimum cardinality is called a $\gamma_{ve}(G)$-set. Vertex-edge domination in graphs was introduced in [7], and further studied in [6].

Chellali and Haynes [1] established the following lower bound on the total domination number of a tree. For every tree T of order n with l leaves we have $\gamma_t(T) \geq (n-l+2)/2$. They also characterized the extremal trees. In [4] a lower bound on the total outer-independent domination number of a tree was given together with

*Department of Mathematics, SASTRA University, Tanjore, Tamilnadu, India

†Research fellow at the Department of Mathematics, University of Johannesburg, South Africa.

‡Faculty of Electronics, Telecommunications and Informatics, Gdansk University of Technology, Poland. Research partially supported by the Polish National Science Centre grant 2011/02/A/ST6/00201.

the characterization of the extremal trees. Lemańska [5] proved that the domination number of a tree is bounded below by $(n - l + 2)/3$.

We prove the following bounds on the vertex-edge domination number of a tree T of order $n \geq 3$ with l leaves and s support vertices, $(n - l - s + 3)/4 \leq \gamma_{ve}(T) \leq n/3$. We also characterize the trees attaining each of the bounds.

2 Results

We begin with the following straightforward observation.

Observation 1 *For every connected graph G of diameter at least two there exists a $\gamma_{ve}(G)$-set that contains no leaf.*

First we show that if T is a nontrivial tree of order n with l leaves and s support vertices, then $\gamma_{ve}(T)$ is bounded below by $(n-l-s+3)/4$. For the purpose of characterizing the trees attaining this bound we introduce a family $\mathcal{T}$ of trees $T = T_k$ that can be obtained as follows. Let T_1 be a path P_5. If k is a positive integer, then T_{k+1} can be obtained recursively from T_k by one of the following operations.

- Operation $\mathcal{O}_1$: Attach a vertex by joining it to any support vertex of T_k.
- Operation $\mathcal{O}_2$: Attach a path P_2 by joining one of its vertices to a vertex of T_k, which is not a leaf and is adjacent to a support vertex of degree two.
- Operation $\mathcal{O}_3$: Attach a path P_4 by joining one of its leaves to a leaf of T_k adjacent to a weak support vertex.

We now prove that for every tree T of the family $\mathcal{T}$ we have $\gamma_{ve}(T) = (n - l - s + 3)/4$.

Lemma 2 *If $T \in \mathcal{T}$, then $\gamma_{ve}(T) = (n - l - s + 3)/4$.*

Proof. We use the induction on the number k of operations performed to construct the tree T. If $T = T_1 = P_5$, then $(n-l-s+3)/4 = (5-2-2+3)/4 = 1 = \gamma_{ve}(T)$. Let k be a positive integer. Assume that the result is true for every tree $T' = T_k$ of the family $\mathcal{T}$ constructed by $k-1$ operations. Let n' be the order of the tree T', l' the number of its leaves, and s' the number of support vertices. Let $T = T_{k+1}$ be a tree of the family $\mathcal{T}$ constructed by k operations.

First assume that T is obtained from T' by operation $\mathcal{O}_1$. We have $n = n' + 1$, $l = l' + 1$ and $s = s'$. It is straightforward to see that any $\gamma_{ve}(T')$-set is a VEDS of the tree T. Thus $\gamma_{ve}(T) \leq \gamma_{ve}(T')$. Obviously, $\gamma_{ve}(T') \leq \gamma_{ve}(T)$. This implies that $\gamma_{ve}(T) = \gamma_{ve}(T')$. We now get $\gamma_{ve}(T) = \gamma_{ve}(T') = (n' - l' - s' + 3)/4 = (n - 1 - l + 1 - s + 3)/4 = (n - l - s + 3)/4$.

Now assume that T is obtained from T' by operation $\mathcal{O}_2$. We have $n = n' + 2$, $l = l' + 1$ and $s = s' + 1$. It is straightforward to see that any $\gamma_{ve}(T')$-set is a VEDS of the tree T. Thus $\gamma_{ve}(T) \leq \gamma_{ve}(T')$. This implies that $\gamma_{ve}(T) = \gamma_{ve}(T')$. We now get $\gamma_{ve}(T) = \gamma_{ve}(T') = (n'-l'-s'+3)/4 = (n-2-l+1-s+1+3)/4 = (n-l-s+3)/4$.

Now assume that T is obtained from T' by operation $\mathcal{O}_3$. We have $n = n'+4$, $l = l'$ and $s = s'$. The leaf to which is attached P_4 we denote by x. Let $v_1v_2v_3v_4$ be the attached path. Let v_1 be joined to x. Let D' be any $\gamma_{ve}(T')$-set. It is easy to see that $D' \cup \{v_2\}$ is a VEDS of the tree T. Thus $\gamma_{ve}(T) \leq \gamma_{ve}(T') + 1$. Now let us observe that there exists a $\gamma_{ve}(T)$-set that does not contain the vertices v_4, v_3 and v_1. Let D be such a set. To dominate the edge v_3v_4, we have $v_2 \in D$. Observe that $D \setminus \{v_2\}$ is a VEDS of the tree T'. Therefore $\gamma_{ve}(T') \leq \gamma_{ve}(T) - 1$. We now conclude that $\gamma_{ve}(T) = \gamma_{ve}(T')+1$. We get $\gamma_{ve}(T) = \gamma_{ve}(T')+1 = (n'-l'-s'+3)/4+1 = (n-4-l-s+7)/4 = (n-l-s+3)/4$. ■

We now give a lower bound on the vertex-edge domination number of a tree together with the characterization of the extremal trees.

Theorem 3 *If T is a nontrivial tree of order n with l leaves and s support vertices, then $\gamma_{ve}(T) \geq (n-l-s+3)/4$ with equality if and only if $T \in \mathcal{T}$.*

Proof. If $\mathrm{diam}(T) = 1$, then $T = P_2$. We have $(n-l-s+3)/4 = (2-2-2+3)/4 < 1 = \gamma_{ve}(T)$. If $\mathrm{diam}(T) = 2$, then T is a star. We have $l = n-1$ and $s = 1$. Consequently, $(n-l-s+3)/4 = (n-n+1-1+3)/4 = 3/4 < 1 = \gamma_{ve}(T)$.

Now assume that $\text{diam}(T) \geq 3$. Thus the order n of the tree T is at least four. The result we obtain by the induction on the number n. Assume that the theorem is true for every tree T' of order $n' < n$ with l' leaves and s' support vertices.

First assume that some support vertex of T, say x, is strong. Let y be a leaf adjacent to x. Let $T' = T - y$. We have $n' = n-1$, $l' = l-1$ and $s' = s$. Obviously, $\gamma_{ve}(T') \leq \gamma_{ve}(T)$. We get $\gamma_{ve}(T) \geq \gamma_{ve}(T') \geq (n'-l'-s'+3)/4 = (n-1-l+1-s+3)/4 = (n-l-s+3)/4$. If $\gamma_{ve}(T) = (n-l-s+3)/4$, then obviously $\gamma_{ve}(T') = (n'-l'-s'+3)/4$. By the inductive hypothesis we have $T' \in \mathcal{T}$. The tree T can be obtained from T' by operation $\mathcal{O}_1$. Thus $T \in \mathcal{T}$. Henceforth, we can assume that every support vertex of T is weak.

We now root T at a vertex r of maximum eccentricity $\text{diam}(T)$. Let t be a leaf at maximum distance from r, v be the parent of t, and u be the parent of v in the rooted tree. If $\text{diam}(T) \geq 4$, then let w be the parent of u. If $\text{diam}(T) \geq 5$, then let d be the parent of w. If $\text{diam}(T) \geq 6$, then let e be the parent of d. By T_x we denote the subtree induced by a vertex x and its descendants in the rooted tree T.

Assume that some child of u, say x, is a leaf. Let $T' = T - x$. We have $n' = n-1$, $l' = l-1$ and $s' = s-1$. We get $\gamma_{ve}(T) \geq \gamma_{ve}(T') \geq (n'-l'-s'+3)/4 = (n-1-l+1-s+1+3)/4 > (n-l-s+3)/4$.

Now assume that among the children of u there is a support vertex other than v. Let $T' = T - T_v$. We have $n' = n-2$, $l' = l-1$ and $s' = s-1$. We get $\gamma_{ve}(T) \geq \gamma_{ve}(T') \geq (n'-l'-s'+3)/4 = (n-2-l+1-s+1+3)/4 = (n-l-s+3)/4$. If $\gamma_{ve}(T) = (n-l-s+3)/4$, then obviously $\gamma_{ve}(T) = (n'-l'-s'+3)/4$. By the inductive hypothesis we have $T' \in \mathcal{T}$. The tree T can be obtained from T' by operation $\mathcal{O}_2$. Thus $T \in \mathcal{T}$.

Now assume that $d_T(u) = 2$. Assume that $d_T(w) \geq 3$. First assume that some child of w, say x, is a leaf. Let $T' = T - x$. We have $n' = n-1$, $l' = l-1$ and $s' = s-1$. We get $\gamma_{ve}(T) \geq \gamma_{ve}(T') \geq (n'-l'-s'+3)/4 = (n-1-l+1-s+1+3)/4 > (n-l-s+3)/4$.

Now assume that no child of w is a leaf. Let $T' = T - T_u$. We have $n' = n-3$, $l' = l-1$ and $s' = s-1$. Let us observe that there exists a $\gamma_{ve}(T)$-set that does not contain the vertices t and v. Let D be such a set. To dominate the edge vt, we have $u \in D$. Let us observe that $D \setminus \{u\}$ is a VEDS of the tree T'. Therefore $\gamma_{ve}(T') \leq \gamma_{ve}(T) - 1$. We now get $\gamma_{ve}(T) \geq \gamma_{ve}(T') + 1 \geq (n'-l'-s'+3)/4 + 1 = (n-3-l+1-s+1+7)/4 > (n-l-s+3)/4$.

If $d_T(w) = 1$, then $T = P_4$. We have $(n-l-s+3)/4 = (4-2-2+3)/4 < 1 = \gamma_{ve}(T)$. Now assume that $d_T(w) = 2$. First assume that $d_T(d) \geq 3$. Let $T' = T - T_w$. We have $n' = n-4$, $l' = l-1$ and $s' = s-1$. Let us observe that there exists a $\gamma_{ve}(T)$-set that does not contain the vertices t, v and w. Let D be such a set. To dominate the edge vt, we have $u \in D$. Observe that $D \setminus \{u\}$ is a VEDS of the tree T'. Therefore $\gamma_{ve}(T') \leq \gamma_{ve}(T) - 1$. We now get $\gamma_{ve}(T) \geq \gamma_{ve}(T') + 1 \geq (n'-l'-s'+3)/4 + 1 = (n-4-l+1-s+1+7)/4 > (n-l-s+3)/4$.

Now assume that $d_T(d) = 2$. First assume that some child of e is a leaf. Let $T' = T - T_w$. We have $n' = n-4$, $l' = l$ and $s' = s-1$. Similarly as in the previous possibility we conclude that $\gamma_{ve}(T') \leq \gamma_{ve}(T) - 1$. We get $\gamma_{ve}(T) \geq \gamma_{ve}(T') + 1 \geq (n'-l'-s'+3)/4 + 1 = (n-4-l-s+1+7)/4 > (n-l-s+3)/4$.

Now assume that no child of e is a leaf. Let $T' = T - T_w$. We have $n' = n-4$, $l' = l$ and $s' = s$. If $n' = 1$, then $T = P_5 = T_1 \in \mathcal{T}$. Assume that $n' \geq 2$. Similarly as earlier we conclude that $\gamma_{ve}(T') \leq \gamma_{ve}(T) - 1$. We now get $\gamma_{ve}(T) \geq \gamma_{ve}(T') + 1 \geq (n'-l'-s'+3)/4 + 1 = (n-4-l-s+7)/4 = (n-l-s+3)/4$. If $\gamma_{ve}(T) = (n-l-s+3)/4$, then obviously $\gamma_{ve}(T') = (n'-l'-s'+3)/4$. By the inductive hypothesis we have $T' \in \mathcal{T}$. The tree T can be obtained from T' by operation $\mathcal{O}_3$. ■

Next we show that if T is a tree of order $n \geq 3$, then $\gamma_{ve}(T)$ is bounded above by $n/3$. For the purpose of characterizing the trees attaining this bound we introduce a family $\mathcal{F}$ of trees $T = T_k$ that can be obtained as follows. Let T_1 be a path P_3. If k is a positive integer, then T_{k+1} can be obtained recursively from T_k by attaching a path P_3 by joining one of its leaves to a vertex of T_k adjacent to a path P_2 or P_3.

We now prove that for every tree T of the family $\mathcal{F}$ we have $\gamma_{ve}(T) = n/3$.

Lemma 4 *If $T \in \mathcal{F}$, then $\gamma_{ve}(T) = n/3$.*

Proof. We use the induction on the number k of operations performed to construct the tree T. If $T = T_1 = P_3$, then $\gamma_{ve}(T) = 1 = n/3$. Let k be a positive integer. Assume that the result is true for every tree $T' = T_k$ of the family $\mathcal{F}$ constructed by $k-1$ operations. Let n' be the order of the tree T'. Let $T = T_{k+1}$ be a tree of the family $\mathcal{F}$ constructed by k operations. We have $n = n' + 3$. The vertex to which is attached P_3 we denote by x. Let $v_1v_2v_3$ be the attached path. Let v_1 be adjacent to x. Let D' be any $\gamma_{ve}(T')$-set. It is easy to see that $D' \cup \{v_1\}$ is a VEDS of the tree T. Thus $\gamma_{ve}(T) \leq \gamma_{ve}(T') + 1$. If x is adjacent to a path P_2, then let us observe that there exists a $\gamma_{ve}(T)$-set that contains the vertices v_1 and x. Let D be such a set. The set D is minimal, thus $v_2, v_3 \notin D$. It is easy to observe that $D \setminus \{v_1\}$ is a VEDS of the tree T'. If x is adjacent to a path P_3 different from $v_1v_2v_3$, say abc, then let a and x be adjacent. Let us observe that there exists a $\gamma_{ve}(T)$-set that contains the vertices v_1 and a. Let D be such a set. The set D is minimal, thus $v_2, v_3 \notin D$. Let us observe that $D \setminus \{v_1\}$

is a VEDS of the tree T'. We now conclude that $\gamma_{ve}(T') \leq \gamma_{ve}(T) - 1$, and consequently, $\gamma_{ve}(T) = \gamma_{ve}(T') + 1$. We get $\gamma_{ve}(T) = \gamma_{ve}(T') + 1 = n'/3 + 1 = (n-3)/3 + 1 = n/3$. ■

We now give an upper bound on the vertex-edge domination number of a tree together with the characterization of the extremal trees.

Theorem 5 *If T is a tree of order $n \geq 3$, then $\gamma_{ve}(T) \leq n/3$ with equality if and only if $T \in \mathcal{F}$.*

Proof. First assume that $\mathrm{diam}(T) = 2$. Thus T is a star. If $T = P_3$, then $T = T_1 \in \mathcal{F}$. If T is a star different from P_3, then we get $n/3 > 1 = \gamma_{ve}(T)$.

Now assume that $\mathrm{diam}(T) \geq 3$. Thus the order n of the tree T is at least four. The result we obtain by the induction on the number n. Assume that the theorem is true for every tree T' of order $n' < n$.

First assume that some support vertex of T, say x, is strong. Let y be a leaf adjacent to x. Let $T' = T - y$. It is straightforward to see that any $\gamma_{ve}(T')$-set is a VEDS of the tree T. Thus $\gamma_{ve}(T) \leq \gamma_{ve}(T')$. We now get $\gamma_{ve}(T) \leq \gamma_{ve}(T') \leq n'/3 < n/3$.

We now root T at a vertex r of maximum eccentricity $\mathrm{diam}(T)$. Let t be a leaf at maximum distance from r, v be the parent of t, and u be the parent of v in the rooted tree. If $\mathrm{diam}(T) \geq 4$, then let w be the parent of u. By T_x we denote the subtree induced by a vertex x and its descendants in the rooted tree T.

Assume that some child of u, say x, is a leaf. Let $T' = T - x$. Let D' be a $\gamma_{ve}(T')$-set that contains no leaf. It is easy to observe that D' is a VEDS of the tree T. Thus $\gamma_{ve}(T) \leq \gamma_{ve}(T')$. We now get $\gamma_{ve}(T) \leq \gamma_{ve}(T') \leq n'/3 < n/3$.

Now assume that among the children of u there is a support vertex other than v. Let $T' = T - T_v$. Let us observe that there exists a $\gamma_{ve}(T')$-set that contains the vertex u. Let D' be such a set. It is easy to see that D' is a VEDS of the tree T. Thus $\gamma_{ve}(T) \leq \gamma_{ve}(T')$. We now get $\gamma_{ve}(T) \leq \gamma_{ve}(T') \leq n'/3 < n/3$.

Now assume that $d_T(u) = 2$. First assume that w is adjacent to a leaf, say x. Let $T' = T - x$. Let us observe that there exists a $\gamma_{ve}(T')$-set that contains the vertex u. Let D' be such a set. It is easy to see that D' is a VEDS of the tree T. Thus $\gamma_{ve}(T) \leq \gamma_{ve}(T')$. We now get $\gamma_{ve}(T) \leq \gamma_{ve}(T') \leq n'/3 < n/3$.

Now assume that there is a child of w other than u, say x, such that the distance of w to the most distant vertex of T_x is two or three. It suffices to consider only the possibilities when T_x is a path P_2 or P_3. Let $T' = T - T_u$. We have $n' = n - 3$. Let D' be any $\gamma_{ve}(T')$-set. It is easy to observe that $D' \cup \{u\}$ is a VEDS of the tree T. Thus $\gamma_{ve}(T) \leq \gamma_{ve}(T') + 1$. We now get $\gamma_{ve}(T) \leq \gamma_{ve}(T') + 1 \leq n'/3 + 1 = n/3$. If $\gamma_{ve}(T) = n/3$, then obviously $\gamma_{ve}(T') = n'/3$. By the inductive hypothesis we have $T' \in \mathcal{F}$. The tree T can be obtained from T' by attaching a path P_3 by joining one of its leaves to the vertex u. Thus $T \in \mathcal{F}$.

Now assume that $d_T(w) = 2$. Let $T' = T - T_w$. We have $n' = n - 4$. If $n' = 1$, then $T = P_5$. We have $\gamma_{ve}(P_5) = 1 < 5/3$. If $n' = 2$, then $T = P_6$. The path P_6 can be obtained from two paths P_3 by joining them through leaves. Thus $T \in \mathcal{F}$. Now assume that $n' \geq 3$. Let D' be any $\gamma_{ve}(T')$-set. Let us observe that $D' \cup \{u\}$ is a VEDS of the tree T. Thus $\gamma_{ve}(T) \leq \gamma_{ve}(T') + 1$. We now get $\gamma_{ve}(T) \leq \gamma_{ve}(T') + 1 \leq n'/3 + 1 = (n-1)/3 < n/3$. ■

References

[1] M. Chellali and T. Haynes, *A note on the total domination number of a tree,* Journal of Combinatorial Mathematics and Combinatorial Computing 58 (2006), 189–193.

[2] T. Haynes, S. Hedetniemi and P. Slater, *Fundamentals of Domination in Graphs,* Marcel Dekker, New York, 1998.

[3] T. Haynes, S. Hedetniemi and P. Slater (eds.), *Domination in Graphs: Advanced Topics,* Marcel Dekker, New York, 1998.

[4] M. Krzywkowski, *A lower bound on the total outer-independent domination number of a tree,* Comptes Rendus Mathematique 349 (2011), 7–9.

[5] M. Lemańska, *Lower bound on the domination number of a tree,* Discussiones Mathematicae Graph Theory 24 (2004), 165–169.

[6] J. Lewis, S. Hedetniemi, T. Haynes and G. Fricke, *Vertex-edge domination,* Utilitas Mathematica 81 (2010), 193–213.

[7] J. Peters, *Theoretical and Algorithmic Results on Domination and Connectivity,* Ph.D. Thesis, Clemson University, 1986.