Total domination stability in graphs

Michael A. Henning* Marcin Krzywkowski†‡

mahenning@uj.ac.za marcin.krzywkowski@gmail.com

Abstract

A set D of vertices in an isolate-free graph G is a total dominating set of G if every vertex is adjacent to a vertex in D. The total domination number of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a total dominating set of G. We note that $\gamma_t(G) \geq 2$ for every isolate-free graph G. A non-isolating set of vertices in G is a set of vertices whose removal from G produces an isolate-free graph. The γ_t^--stability of G, denoted by $st_{\gamma_t}^-(G)$, is the minimum size of a non-isolating set of vertices in G whose removal decreases the total domination number. We show that if G is a connected graph with maximum degree Δ satisfying $\gamma_t(G) \geq 3$, then $st_{\gamma_t}^-(G) \leq 2\Delta - 1$, and we characterize the infinite family of trees achieving the equality in this upper bound. The total domination stability of G, denoted by $st_{\gamma_t}(G)$, is the minimum size of a non-isolating set of vertices in G whose removal changes the total domination number. We prove that if G is a connected graph with maximum degree Δ satisfying $\gamma_t(G) \geq 3$, then $st_{\gamma_t}(G) \leq 2\Delta - 1$.

Keywords: total domination, total domination stability.
$\mathcal{AMS}$ **Subject Classification:** 05C05, 05C69.

1 Introduction

The concept of domination stability in graphs was introduced in 1983 by Bauer, Harary, Nieminen and Suffel [1] and has been studied, for example, in [13]. We

*Department of Pure and Applied Mathematics, University of Johannesburg, South Africa. Research supported in part by the South African National Research Foundation and the University of Johannesburg.

†Research fellow of the Claude Leon Foundation at the Department of Pure and Applied Mathematics, University of Johannesburg, South Africa.

‡Faculty of Electronics, Telecommunications and Informatics, Gdansk University of Technology, Poland. Research partially supported by the Polish National Science Centre grant 2011/02/A/ST6/00201.

introduce and study the total version of domination stability. We demonstrate that these two versions differ significantly.

A *dominating set* of a graph G with vertex set $V(G)$ is a set D of vertices of G such that every vertex in $V(G) \setminus D$ is adjacent to a vertex in D. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. The concept of domination critical graphs is well studied in the literature (see, for example, [1, 2, 3, 4, 6, 15, 16]). We focus on domination stability in graphs. As defined in [1], the γ^--*stability* of G, denoted by $\gamma^-(G)$, is the minimum number of vertices whose removal decreases the domination number, and the γ^+-*stability* of G, denoted by $\gamma^+(G)$, is the minimum number of vertices whose removal increases the domination number. The domination stability of G, denoted by $st_\gamma(G)$, is the minimum number of vertices whose removal changes the domination number. Thus, $st_\gamma(G) = \min\{\gamma^-(G), \gamma^+(G)\}$.

An *isolate-free graph* is a graph with no isolated vertex. A *total dominating set*, abbreviated TD-set, of an isolate-free graph G is a set D of vertices of G such that every vertex in $V(G)$ is adjacent to at least one vertex in D. The *total domination number* of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a TD-set of G. A *non-isolating set* of vertices in G is a set S of vertices such that $G - S$ is an isolate-free graph, where $G - S$ denotes the graph obtained from G by removing S and all edges incident with vertices in S. Let $\mathrm{NI}(G)$ denote the set of all non-isolating sets of vertices of G. The concept of total domination critical graphs is well studied in the literature (see, for example, [5, 7, 10, 11, 17].) Chapter 11 in the book [12] is devoted to total domination critical graphs.

Unless otherwise stated, let G be an isolate-free graph. The γ_t^--stability of G, denoted by $st^-_{\gamma_t}(G)$, is the minimum size of a non-isolating set of vertices in G whose removal decreases the total domination number. Thus,

$$st^-_{\gamma_t}(G) = \min_{S \in \mathrm{NI}(G)} \{|S| \colon \gamma_t(G - S) < \gamma_t(G)\}.$$

The γ_t^+-stability of G, denoted by $st^+_{\gamma_t}(G)$, is the minimum size of a non-isolating set of vertices in G whose removal increases the total domination number, if such a set exists. In this case,

$$st^+_{\gamma_t}(G) = \min_{S \in \mathrm{NI}(G)} \{|S| \colon \gamma_t(G - S) > \gamma_t(G)\}.$$

If no such non-isolating set exists whose removal increases the total domination number, we define $st^+_{\gamma_t}(G) = \infty$. As a trivial example, we have $st^-_{\gamma_t}(P_7) = 2$ while $st^+_{\gamma_t}(P_7) = \infty$.

The *total domination stability* of G, denoted by $st_{\gamma_t}(G)$, is the minimum size of a non-isolating set of vertices in G whose removal changes the total domination number. Thus,

$$st_{\gamma_t}(G) = \min_{S \in \mathrm{NI}(G)} \{|S| \colon \gamma_t(G - S) \neq \gamma_t(G)\} = \min\{st^-_{\gamma_t}(G), st^+_{\gamma_t}(G)\}.$$

2 Preliminaries

For notation and graph theory terminology we generally follow [12]. The *order* of G is denoted by $n(G) = |V(G)|$, and the *size* of G is denoted by $m(G) = |E(G)|$. We denote the *degree* of a vertex v in the graph G by $d_G(v)$. A vertex of degree 0 is called an *isolated vertex*. The maximum (minimum) degree among the vertices of G is denoted by $\Delta(G)$ ($\delta(G)$, respectively). The *open neighborhood* of v is $N_G(v) = \{u \in V(G)\colon uv \in E(G)\}$ and the *closed neighborhood of* v is $N_G[v] = N_G(v) \cup \{v\}$. For a set $S \subseteq V$, its *open neighborhood* is the set $N_G(S) = \bigcup_{v\in S} N_G(v)$, and its *closed neighborhood* is the set $N_G[S] = N_G(S) \cup S$. If the graph G is clear from the context, then we simply write $d(v)$, $N(v)$, $N[v]$, $N(S)$ and $N[S]$ instead of $d_G(v)$, $N_G(v)$, $N_G[v]$, $N_G(S)$ and $N_G[S]$, respectively.

For a subset S of vertices of G, the subgraph induced by S is denoted by $G[S]$. The subgraph obtained from G by removing all vertices in S and all edges incident with vertices in S is denoted by $G - S$. The set S is an *open packing* if the open neighborhoods of vertices in S are pairwise disjoint. The *open packing number* of G, denoted by $\rho^0(G)$, is the maximum cardinality of an open packing in G.

A *non-trivial graph* is a graph of order at least 1. A path and a cycle on n vertices are denoted by P_n and C_n, respectively. A complete graph on n vertices is denoted by K_n, while a complete bipartite graph with partite sets of size l and m is denoted by $K_{l,m}$. A *star* is the graph $K_{1,k}$, where $k \ge 1$. For $r, s \ge 1$, a *double star* $S(r, s)$ is the tree with exactly two vertices that are not leaves, one of which has r leaf-neighbors and the other s leaf-neighbors.

A *rooted tree* T distinguishes one vertex r called the *root*. For each vertex $v \neq r$ of T, the *parent* of v is the neighbor of v on the unique (r, v)-path, while a *child* of v is any other neighbor of v. The set of children of v is denoted by $C(v)$. A *descendant* of v is a vertex $u \neq v$ such that the unique (r, u)-path contains v, while an *ancestor* of v is a vertex $u \neq v$ that belongs to the (r, v)-path in T. In particular, every child of v is a descendant of v, while the parent of v is an ancestor of v.

The *distance* between two vertices u and v in a connected graph G, denoted by $d_G(u, v)$, is the length of a shortest (u, v)-path in G. The maximum distance among all pairs of vertices of G is the *diameter* of G, denoted by $\mathrm{diam}(G)$. We use the notation $[k] = \{1, \dots, k\}$.

Following the original paper of Bauer et al. [1], we consider the *null graph* K_0 (also called the *order-zero graph*), which is the unique graph having no vertices and hence has order zero, as a graph. With this consideration, the domination stability of a non-trivial graph is always defined. In particular, $st_\gamma(K_n) = n$ since $\gamma(K_n) = 1$ and removing all vertices from the complete graph on n vertices produces the null graph with domination number zero. Bauer et al. [1] established the following fundamental upper bound on the domination stability of a graph.

Theorem 1 ([1]) *For every nontrivial graph G we have $st_\gamma(G) \le \delta(G) + 1$.*

As an immediate consequence of Theorem 1, we have the following result observed by Jafari Rad et al. [13].

Observation 2 ([13]) *If $G \not\cong K_n$ is a graph of order n, then $st_\gamma(G) \le n - 1$.*

Considering the null graph K_0 as a graph, we have the following observation.

Observation 3 *If G is a graph of order n and $\gamma_t(G) = 2$, then $st^-_{\gamma_t}(G) = n$.*

In view of Observation 3, it is only of interest for us to consider isolate-free graphs G with $\gamma_t(G) \ge 3$ when determining $st^-_{\gamma_t}(G)$. With this assumption, we note that if u and v are adjacent vertices in G, then the set $S = V(G) \setminus \{u, v\}$ is a non-isolating set of vertices in G and $\gamma_t(G - S) = \gamma_t(K_2) = 2 < \gamma_t(G)$. Thus, the γ_t^--stability of an isolate-free graph G with $\gamma_t(G) \ge 3$ is at most two less than its order. We state this formally as follows.

Observation 4 *If G is a graph of order n and $\gamma_t(G) \ge 3$, then $st^-_{\gamma_t}(G) \le n - 2$.*

Rall [14] was the first to prove that the total domination number and the open packing number of any non-trivial tree are equal.

Theorem 5 ([14]) *For every non-trivial tree T we have $\gamma_t(T) = \rho^o(T)$.*

3 Main results

Our immediate aim is to establish upper bounds on the total domination stability and the γ_t^--stability of a graph. We first establish an upper bound on the γ_t^--stability of a tree. Further, we characterize the trees with maximum possible γ_t^--stability. For this purpose, we define a family of trees as follows.

For integers $k \ge 2$ and $\Delta \ge 2$, let $T_{k,\Delta}$ be a graph obtained from the disjoint union of k double stars $S(\Delta - 1, \Delta - 1)$ by adding $k - 1$ edges between the leaves of these double stars so that the resulting graph is a tree with maximum degree Δ. Let $\mathcal{F}_{k,\Delta}$ be the family of all such trees $T_{k,\Delta}$, and let

$$\mathcal{F}_\Delta = \bigcup_{k \ge 2} \mathcal{F}_{k,\Delta}.$$

For $\Delta = 2$, let $\mathcal{H}_\Delta = \{P_n : n \equiv 3 \pmod 4 \text{ and } n \ge 7\}$. For integers $\Delta \ge 3$ and $\Delta \ge k \ge 2$, let $H_{k,\Delta}$ be a graph obtained from the disjoint union of k double stars $S(\Delta - 1, \Delta - 1)$ by selecting one leaf from each double star and identifying these k leaves into one new vertex. Let $\mathcal{H}_{k,\Delta}$ be the family of all such trees $H_{k,\Delta}$, and let

$$\mathcal{H}_\Delta = \bigcup_{k \ge 2} \mathcal{H}_{k,\Delta}.$$

We shall prove the following result.

Theorem 6 *If T is a tree with maximum degree Δ satisfying $\gamma_t(T) \geq 3$, then the following hold:*

(a) $st^-_{\gamma_t}(T) \leq 2\Delta - 1$, with equality if and only if $T \in \mathcal{F}_\Delta$;

(b) $st_{\gamma_t}(T) \leq 2\Delta - 2$, and this bound is sharp for all $\Delta \geq 2$.

Recall that by Theorem 1, for every nontrivial graph G, we have $st_\gamma(G) \leq \delta(G)+1$. In particular, $st_\gamma(T) \leq 2$ for every nontrivial tree T. This is in contrast to the total domination stability, where for any given $\Delta \geq 2$, every tree T in the family $\mathcal{H}_\Delta$ has maximum degree Δ and satisfies $st_{\gamma_t}(T) = 2\Delta - 2$ (as shown in Lemma 16). Thus, total domination stability differs significantly from domination stability.

The following result establishes an upper bounds on the total domination stability of a general graph in terms of its maximum degree.

Theorem 7 *If G is a connected graph with maximum degree Δ satisfying $\gamma_t(G) \geq 3$, then $st_{\gamma_t}(G) \leq 2\Delta - 1$.*

4 Preliminary results

It is well known (see, for example, [12]) that $\gamma_t(C_n) = \gamma_t(P_n) = \lfloor n/2 \rfloor + \lceil n/4 \rceil - \lfloor n/4 \rfloor$ for all $n \geq 3$. We first determine the γ_t^--stability of a path and a cycle.

Proposition 8 *For $n \geq 5$, if G is a path P_n or a cycle C_n, then*

$$st^-_{\gamma_t}(G) = \begin{cases} 3 & \text{when } n \equiv 0 \pmod{4} \\ 2 & \text{when } n \equiv 3 \pmod{4} \\ 1 & \text{when } n \equiv 1,2 \pmod{4}. \end{cases}$$

Proof. For $n \geq 5$, if $G \cong P_n$, then let G be given by $v_1v_2\dots v_n$, while if $G \cong C_n$, then let G be given by $v_1v_2\dots v_nv_1$. Suppose first that $n \equiv 2 \pmod 4$. Thus, $n = 4k+2$ for some $k \geq 1$. In this case, $\gamma_t(G) = 2k+2$. Taking $S = \{v_1\}$, we note that $\gamma_t(G-S) = \gamma_t(P_{4k+1}) = 2k+1 < \gamma_t(G)$, implying that $st^-_{\gamma_t}(G) = 1$.

Suppose second that $n \equiv 1 \pmod 4$. Thus, $n = 4k+1$ for some $k \geq 1$. In this case, $\gamma_t(G) = 2k+1$. Taking $S = \{v_1\}$, we note that $\gamma_t(G-S) = \gamma_t(P_{4k}) = 2k < \gamma_t(G)$, implying that $st^-_{\gamma_t}(G) = 1$.

Suppose next that $n \equiv 0 \pmod 4$. Thus, $n = 4k$ for some $k \geq 2$. In this case, $\gamma_t(G) = 2k$. Let S be a non-isolating set of vertices such that $\gamma_t(G-S) \leq 2k-1$. Let D be a minimum TD-set of $G-S$, and so $|D| = \gamma_t(G-S) \leq 2k-1$. Suppose that $G[D]$ consists of ℓ components. Each component of $G[D]$ is a path of order at least 2, implying that $2k-1 \geq |D| \geq 2\ell$ and therefore that $\ell \leq k-1$. Further if P is a (path) component of $G[D]$, then each end of P is adjacent to at most one vertex of $G-S$ that does not belong to P. Thus, at most two vertices of $G-S$ that do not belong to the set D are uniquely associated with each component of

$G[D]$, implying that $G-S$ has order at most $|D|+2\ell \le (2k-1)+2(k-1) = 4k-3$. However, $G-S$ has order $4k-|S|$, and so $|S| \ge 3$. This is true for every non-isolating set S of vertices of G. Hence, $st^-_{\gamma_t}(G) \ge 3$. Conversely, if we take $S^* = \{v_1, v_2, v_3\}$, then $\gamma_t(G-S) = \gamma_t(P_{4k-3}) = 2k-1$, and so $st^-_{\gamma_t}(G) \le |S^*| = 3$. Consequently, $st^-_{\gamma_t}(G) = 3$.

Suppose finally that $n \equiv 3 \pmod 4$. Thus, $n = 4k-1$ for some $k \ge 2$. In this case, $\gamma_t(G) = 2k$. Let S be a non-isolating set of vertices such that $\gamma_t(G-S) \le 2k-1$. Proceeding analogously as in the previous case, we show that $G-S$ has order at most $4k-3$. Since $G-S$ has order $4k-1-|S|$, we deduce that $|S| \ge 2$, implying that $st^-_{\gamma_t}(G) \ge 2$. Conversely, if we take $S^* = \{v_1, v_2\}$, then $\gamma_t(G-S) = \gamma_t(P_{4k-3}) = 2k-1$, and so $st^-_{\gamma_t}(G) \le |S^*| = 2$. Consequently, $st^-_{\gamma_t}(G) = 2$. ■

Next we determine the γ_t^+-stability of a path. For small $n \le 7$ and for $n = 10$, no non-isolating set of vertices in a path P_n exists whose removal increases the total domination number, and hence, by definition, $st^+_{\gamma_t}(P_n) = \infty$ for such values of n. It is therefore only of interest to determine the γ_t^+-stability of a path P_n, where $n \ge 8$ and $n \ne 10$.

Proposition 9 *For $n \ge 8$ and $n \ne 10$,*

$$st^+_{\gamma_t}(P_n) = \begin{cases} 3 & \text{when } n \equiv 2 \pmod 4; \\ 2 & \text{when } n \equiv 3 \pmod 4; \\ 1 & \text{when } n \equiv 0, 1 \pmod 4. \end{cases}$$

Proof. Consider a path $G \cong P_n$ given by $v_1 v_2 \ldots v_n$, where $n \ge 8$. Suppose first that $n \equiv 3 \pmod 4$. Thus, $n = 4k-1$ for some $k \ge 2$. In this case, $\gamma_t(G) = 2k$. Taking $S = \{v_3, v_6\}$, we note that S is a non-isolating set of vertices and $\gamma_t(G-S) = 2\gamma_t(P_2) + \gamma_t(P_{4k-7}) = 4 + (2k-3) = 2k+1 > \gamma_t(G)$, implying that $st^+_{\gamma_t}(P_n) \le |S| = 2$. Conversely, suppose that S is a non-isolating set consisting of a single vertex. In this case, $G-S$ contains at most two components. If $G-S$ is connected, then $\gamma_t(G-S) = \gamma_t(P_{4k-2}) = 2k \le \gamma_t(G)$. Suppose that $G-S$ contains two components. Let P_{n_1} and P_{n_2} be the two path components of $G-S$, and so $\gamma_t(G-S) = \gamma_t(P_{n_1}) + \gamma_t(P_{n_2})$. Since $n_1 + n_2 = 4k-2$, either both n_1 and n_2 are even or both n_1 and n_2 are odd. If both n_1 and n_2 are even, then we may assume that $n_1 \equiv 0 \pmod 4$ and $n_2 \equiv 2 \pmod 4$. In this case, $\gamma_t(G-S) = n_1/2 + (n_2+2)/2 = 2k$. If both n_1 and n_2 are odd, then $\gamma_t(G-S) = (n_1+1)/2 + (n_2+1)/2 = 2k$. In both cases, $\gamma_t(G-S) = 2k = \gamma_t(G)$, implying that $st^+_{\gamma_t}(P_n) > 1$. Consequently, $st^+_{\gamma_t}(P_n) = 2$.

Suppose second that $n \equiv 0 \pmod 4$. Thus, $n = 4k$ for some $k \ge 2$. In this case, $\gamma_t(G) = 2k$. Taking $S = \{v_3\}$, we note that S is a non-isolating set of vertices such that $\gamma_t(G-S) = \gamma_t(P_2) + \gamma_t(P_{4k-3}) = 2 + (2k-1) = 2k+1 > \gamma_t(G)$, implying that $st^+_{\gamma_t}(P_n) \le |S| = 1$. Consequently, $st^+_{\gamma_t}(P_n) = 1$.

Suppose next that $n \equiv 1 \pmod 4$. Thus, $n = 4k + 1$ for some $k \geq 2$. In this case, $\gamma_t(G) = 2k + 1$. Taking $S = \{v_3\}$, we note that $\gamma_t(G - S) = \gamma_t(P_2) + \gamma_t(P_{4k-2}) = 2 + 2k > \gamma_t(G)$, implying that $st^+_{\gamma_t}(P_n) = 1$.

Suppose finally that $n \equiv 2 \pmod 4$. Thus, $n = 4k + 2$ for some $k \geq 3$. In this case, $\gamma_t(G) = 2k + 2$. Suppose that S is a non-isolating set consisting of a single vertex. In this case, $G - S$ contains at most two components. If $G - S$ is connected, then $\gamma_t(G - S) = \gamma_t(P_{4k+1}) = 2k + 1 \leq \gamma_t(G)$. Suppose that $G - S$ contains two components. Let P_{n_1} and P_{n_2} be the two path components of $G - S$, and so $\gamma_t(G - S) = \gamma_t(P_{n_1}) + \gamma_t(P_{n_2})$. Since $n_1 + n_2 = 4k + 1$, exactly one of n_1 and n_2 is odd, say n_1. Thus, $\gamma_t(G - S) \leq (n_1 + 1)/2 + (n_2 + 2)/2 = 2k + 2 = \gamma_t(G)$.

Suppose that S is a non-isolating set of size 2. In this case, $G - S$ contains at most three components. If $G - S$ is connected, then $\gamma_t(G - S) = \gamma_t(P_{4k}) = 2k < \gamma_t(G)$.

Suppose that $G - S$ contains two components. Let P_{n_1} and P_{n_2} be the two path components of $G - S$, and so $\gamma_t(G - S) = \gamma_t(P_{n_1}) + \gamma_t(P_{n_2})$. Since $n_1 + n_2 = 4k$, either both n_1 and n_2 are even or both n_1 and n_2 are odd. If both n_1 and n_2 are even, then $\gamma_t(G - S) \leq (n_1 + 2)/2 + (n_2 + 2)/2 = 2k + 2$. If both n_1 and n_2 are odd, then $\gamma_t(G - S) = (n_1 + 1)/2 + (n_2 + 1)/2 = 2k + 1$. In both cases, $\gamma_t(G - S) \leq \gamma_t(G)$.

Suppose that $G - S$ contains three components. Let P_{n_1}, P_{n_2} and P_{n_3} be the three path components of $G - S$, and so $\gamma_t(G - S) = \gamma_t(P_{n_1}) + \gamma_t(P_{n_2}) + \gamma_t(P_{n_3})$. We note that $n_1 + n_2 + n_3 = 4k$. A straightforward case analysis shows that, renaming n_1, n_2 and n_3 if necessary, one of the following cases occur: (i) $n_i \equiv 0 \pmod 4$ for $i \in [3]$, (ii) $n_i \equiv 2 \pmod 4$ for $i \in [2]$ and $n_3 \equiv 0 \pmod 4$, (iii) $n_1 \equiv 0 \pmod 4$, $n_2 \equiv 1 \pmod 4$, $n_3 \equiv 3 \pmod 4$, (iv) $n_i \equiv 1 \pmod 4$ for $i \in [2]$ and $n_3 \equiv 2 \pmod 4$, (v) $n_i \equiv 3 \pmod 4$ for $i \in [2]$ and $n_3 \equiv 2 \pmod 4$. If (i) holds, then $\gamma_t(G - S) = 2k$. If (ii) holds, then $\gamma_t(G - S) = 2k + 2$. If (iii) holds, then $\gamma_t(G - S) = 2k + 1$. If (iv) or (v) holds, then $\gamma_t(G - S) = 2k + 2$. In all five cases, $\gamma_t(G - S) \leq \gamma_t(G)$.

The above implies that if S is an arbitrary non-isolating set of size 1 or 2, then $\gamma_t(G - S) \leq \gamma_t(G)$, implying that $st^+_{\gamma_t}(P_n) \geq 3$. Conversely, taking $S = \{v_3, v_6, v_9\}$, we note that S is a non-isolating set of vertices and $\gamma_t(G - S) = 3\gamma_t(P_2) + \gamma_t(P_{4k-7}) = 6 + (2k - 3) = 2k + 3 > \gamma_t(G)$, implying that $st^+_{\gamma_t}(P_n) \leq |S| = 3$. Consequently, $st^+_{\gamma_t}(P_n) = 3$. ■

As an immediate consequence of Propositions 8 and 9, we determine the total domination stability of a path.

Proposition 10 *For $n \geq 5$,*

$$st_{\gamma_t}(P_n) = \begin{cases} 2 & \textit{when } n \equiv 3 \pmod 4; \\ 1 & \textit{otherwise.} \end{cases}$$

As shown in Proposition 8, the γ_t^--stability of a path and a cycle of the same order are equal. This is not always the case for the γ_t^+-stability of a path and a cycle. For example, $st^+_{\gamma_t}(P_8) = 1$ and $st^+_{\gamma_t}(P_{14}) = 3$, while $st^+_{\gamma_t}(C_8) = st^+_{\gamma_t}(C_{14}) = \infty$. For small $n \le 8$ and for $n \in \{10, 11, 14\}$, no non-isolating set of vertices in a cycle C_n exists whose removal increases the total domination number, and hence, by definition, $st^+_{\gamma_t}(C_n) = \infty$ for such values of n. In order for us to determine the total domination stability of a cycle when $n \ge 5$, it suffices for us to establish the following result on the γ_t^+-stability of a cycle.[1]

Proposition 11 *For $n \ge 9$ and $n \notin \{10, 11, 14\}$, $st^+_{\gamma_t}(C_n) \ge 3$. Further, the following hold.*
(a) If $n \ge 9$ and $n \equiv 1 \pmod 4$, then $st^+_{\gamma_t}(C_n) = 3$;
(b) If $n \ge 12$ and $n \equiv 0 \pmod 4$, then $st^+_{\gamma_t}(C_n) = 3$.

Proof. Consider a cycle $G \cong C_n$ given by $v_1v_2 \dots v_nv_1$, where $n \ge 9$ and $n \notin \{10, 11, 14\}$. Let S be a non-isolating set of vertices of G that increases the total domination number. If $|S| = 1$, then $\gamma_t(G - S) = \gamma_t(P_{n-1}) \le \gamma_t(P_n) = \gamma_t(G)$, a contradiction. Hence, $|S| \ge 2$. Suppose that $|S| = 2$. In this case, $G - S$ contains at most two components. If $G - S$ is connected, then $\gamma_t(G - S) = \gamma_t(P_{n-2}) \le \gamma_t(P_n) = \gamma_t(G)$, a contradiction. Hence, $G - S$ contains two components. Let P_{n_1} and P_{n_2} be the two path components of $G - S$, and so $\gamma_t(G - S) = \gamma_t(P_{n_1}) + \gamma_t(P_{n_2})$.

Suppose that $n \equiv 0 \pmod 4$. Thus, $n = 4k$ for some $k \ge 3$. In this case, $\gamma_t(G) = 2k$. Since $n_1 + n_2 = 4k - 2$, renaming n_1 and n_2 if necessary, either $n_1 \equiv 0 \pmod 4$ and $n_2 \equiv 2 \pmod 4$ or both n_1 and n_2 are odd. In the former case, $\gamma_t(G - S) = n_1/2 + (n_2 + 2)/2 = 2k$, while in the latter case, $\gamma_t(G - S) = (n_1 + 1)/2 + (n_2 + 1)/2 = 2k$. In both cases, $\gamma_t(G - S) \le \gamma_t(G)$, a contradiction.

Suppose that $n \equiv 1 \pmod 4$. Thus, $n = 4k + 1$ for some $k \ge 2$. In this case, $\gamma_t(G) = 2k + 1$. Since $n_1 + n_2 = 4k - 1$, exactly one of n_1 and n_2 is odd, say n_1. Thus, $\gamma_t(G - S) \le (n_1 + 1)/2 + (n_2 + 2)/2 = 2k + 1 = \gamma_t(G)$, a contradiction.

Suppose that $n \equiv 2 \pmod 4$. Thus, $n = 4k + 2$ for some $k \ge 4$. In this case, $\gamma_t(G) = 2k + 2$. Since $n_1 + n_2 = 4k$, either both n_1 and n_2 are even or both n_1 and n_2 are odd. If both n_1 and n_2 are even, then $\gamma_t(G - S) \le (n_1 + 2)/2 + (n_2 + 2)/2 = 2k + 2$. If both n_1 and n_2 are odd, then $\gamma_t(G - S) = (n_1 + 1)/2 + (n_2 + 1)/2 = 2k + 1$. In both cases, $\gamma_t(G - S) \le \gamma_t(G)$, a contradiction.

Suppose that $n \equiv 3 \pmod 4$. Thus, $n = 4k + 3$ for some $k \ge 3$. In this case, $\gamma_t(G) = 2k + 2$. Since $n_1 + n_2 = 4k + 1$, exactly one of n_1 and n_2 is odd, say n_1. Thus, $\gamma_t(G - S) \le (n_1 + 1)/2 + (n_2 + 2)/2 = 2k + 2 = \gamma_t(G)$, a contradiction. Since all the above four cases produce a contradiction, every non-isolating set S of vertices of G that increases the total domination number satisfies $|S| \ge 3$, implying that $st^+_{\gamma_t}(G) \ge 3$.

[1]We remark that the result of Proposition 11 can be strengthened to cover all values of $n \ge 9$ and $n \notin \{10, 11, 14\}$. Indeed, if $n \ge 15$ and $n \equiv 3 \pmod 4$, then $st^+_{\gamma_t}(C_n) = 4$, while if $n \ge 18$ and $n \equiv 2 \pmod 4$, then $st^+_{\gamma_t}(C_n) = 5$. We omit the details.

Suppose that $n \geq 9$ and $n \equiv 1 \pmod 4$. Thus, $n = 4k + 1$ for some $k \geq 2$. In this case, $\gamma_t(G) = 2k+1$. Taking $S = \{v_1, v_4, v_7\}$, we note that S is a non-isolating set of vertices and $\gamma_t(G-S) = 2\gamma_t(P_2)+\gamma_t(P_{4k-6}) = 4+(2k-2) = 2k+2 > \gamma_t(G)$, implying that $st^+_{\gamma_t}(C_n) \leq |S| = 3$. Consequently, $st^+_{\gamma_t}(C_n) = 3$.

Suppose that $n \geq 12$ and $n \equiv 0 \pmod 4$. Thus, $n = 4k$ for some $k \geq 3$. In this case, $\gamma_t(G) = 2k$. Taking $S = \{v_1, v_4, v_7\}$, we note that S is a non-isolating set of vertices and $\gamma_t(G-S) = 2\gamma_t(P_2)+\gamma_t(P_{4k-7}) = 4+(2k-3) = 2k+1 > \gamma_t(G)$, implying that $st^+_{\gamma_t}(C_n) \leq |S| = 3$. Consequently, $st^+_{\gamma_t}(C_n) = 3$. ■

As a consequence of Propositions 8 and 11, the total domination stability of a cycle is determined.

Proposition 12 *For $n \geq 5$,*

$$st_{\gamma_t}(C_n) = \begin{cases} 3 & \textit{when } n \equiv 0 \pmod 4; \\ 2 & \textit{when } n \equiv 3 \pmod 4; \\ 1 & \textit{when } n \equiv 1, 2 \pmod 4. \end{cases}$$

We shall need the following lemma.

Lemma 13 *If G is a connected isolate-free graph, then G contains a spanning tree T such that $\gamma_t(T) = \gamma_t(G)$.*

Proof. Since removing edges from a graph cannot decrease its total domination number, we note that $\gamma_t(H) \geq \gamma_t(G)$ for all isolate-free spanning subgraphs H of G. In particular, if $H = G$, then $\gamma_t(H) = \gamma_t(G)$. Among all spanning, isolate-free, connected subgraphs H of G satisfying $\gamma_t(H) = \gamma_t(G)$, let H be chosen to have minimum size. Let S be an arbitrary minimum TD-set of H.

We show that H is a tree. Suppose, to the contrary, that H is not a tree and consider a smallest cycle $C\colon v_1v_2 \ldots v_kv_1$ in H. For $i \in [k]$, let $e_i = v_iv_{i+1}$, where addition is taken modulo k, and so $e_k = v_1v_k$.

By the minimality of H, the spanning, isolate-free, connected graph $H - e_i$ satisfies $\gamma_t(H - e_i) > \gamma_t(G)$ for each $i \in [k]$. If neither v_i nor v_{i+1} belong to S, then S is a TD-set of $H - e_i$, implying that $\gamma_t(G) \leq \gamma_t(H - e_i) \leq |S| = \gamma_t(G)$, or, equivalently, $\gamma_t(H - e_i) = \gamma_t(G)$, a contradiction. Hence, the set S contains at least one of v_i and v_{i+1} for all $i \in [k]$. If S contains all vertices on the cycle C, then S is a TD-set of $H - e_i$ for all $i \in [k]$, a contradiction. Hence, renaming vertices if necessary, we may assume that $v_2 \notin S$. Thus, by our earlier observations, $v_1 \in S$ and $v_3 \in S$. But then S is a TD-set of $H - e_1$, a contradiction. ■

We determine next the γ_t^--stability of a tree in the family $\mathcal{F}_\Delta$.

Lemma 14 *For $\Delta \geq 2$, if $T \in \mathcal{F}_\Delta$, then $st^-_{\gamma_t}(T) = 2\Delta - 1$.*

Proof. For $k \geq 2$ and $\Delta \geq 2$, let $T \in \mathcal{F}_{k,\Delta}$. We wish to show that $st^{-}_{\gamma_t}(T) = 2\Delta - 1$. If $\Delta = 2$, then the family $\mathcal{F}_{k,\Delta}$ is the family of all paths P_{4k}, where $k \geq 2$. Thus in this case, by Proposition 8, $st^{-}_{\gamma_t}(T) = 3 = 2\Delta - 1$. Hence, in what follows we may assume that $\Delta \geq 3$, for otherwise the desired result follows. We proceed by induction on $k \geq 2$ to show that if $T \in \mathcal{F}_{k,\Delta}$, then $st^{-}_{\gamma_t}(T) = 2\Delta - 1$.

For the base case, suppose that $k = 2$, and so $T \in \mathcal{F}_{2,\Delta}$. Thus, T is obtained from the disjoint union of two double stars T_1 and T_2, both isomorphic to $S(\Delta - 1, \Delta - 1)$, by adding and edge joining a leaf, x_1 say, of T_1 and a leaf, x_2 say, of T_2. Let u_i and v_i be the two central vertices of the double star T_i for $i \in [2]$, where vx_i is an edge, and so $u_iv_ix_i$ is a path in T_i. The set $\{u_1, u_2, v_1, v_2\}$ is a maximum open packing, and so, by Theorem 5, $\gamma_t(T) = \rho^0(T) = 4$. We note that $\text{diam}(T) = 7$.

Let S be a non-isolating set of vertices of T such that $\gamma_t(T - S) \leq 3$. We show that $|S| \geq 2\Delta - 1$. Let D be a minimum TD-set of $T - S$, and so $|D| = \gamma_t(T - S) \leq 3$. We note that the set D induces either a path P_2 or a path P_3, implying that $T - S$ is a tree and $\text{diam}(T - S) \leq 4$. If both u_1 and u_2 belong to the tree $T - S$, then $\text{diam}(T - S) \geq 5$, a contradiction. Hence, renaming T_1 and T_2, if necessary, we may assume that $u_1 \in S$. If $u_2 \in S$, then S contains all leaf-neighbors of u_1 and u_2, implying that $|S| \geq 2\Delta$. Hence, we may assume that $u_2 \notin S$, and so u_2 belongs to the tree $T - S$. If $v_1 \notin S$, then since $\text{diam}(T - S) \leq 4$, the set S contains all leaf-neighbors of v_1 and u_2, implying that $|S| \geq 3\Delta - 3$. Hence, we may assume that $v_1 \in S$, for otherwise $|S| \geq 2\Delta - 1$, as desired. In this case, $V(T_1) \setminus \{x_1\} \subseteq S$, implying that $|S| \geq 2\Delta - 1$. Since S is an arbitrary non-isolating set of vertices of T such that $\gamma_t(T - S) \leq 3$, this implies that $st^{-}_{\gamma_t}(T) \geq 2\Delta - 1$. Conversely, if we take $S = V(T_1) \setminus \{x_1\}$, then $|S| = 2\Delta - 1$ and $\gamma_t(T - S) = 3$, and so $st^{-}_{\gamma_t}(T) \leq 2\Delta - 1$. Consequently, $st^{-}_{\gamma_t}(T) = 2\Delta - 1$. This establishes the base case when $k = 2$.

Let $k \geq 3$ and assume that if $T' \in \mathcal{F}_{k',\Delta}$ where $2 \leq k' < k$, then $st^{-}_{\gamma_t}(T') = 2\Delta - 1$. Let $T \in \mathcal{F}_{k,\Delta}$. Thus, T is the graph obtained from the disjoint union of k double stars $T_1, T_2, \ldots, T_k$, each isomorphic to $S(\Delta - 1, \Delta - 1)$, by adding $k - 1$ edges between leaves of these double stars so that the resulting graph is a tree with maximum degree Δ. Let u_i and v_i be the two central vertices of the double star T_i for $i \in [k]$. The set $\cup_{i=1}^{k}\{u_i, v_i\}$, for example, is a maximum open packing of T, and so, by Theorem 5, $\gamma_t(T) = \rho^0(T) = 2k$.

We define the underlying graph of T as the graph of order k whose vertices correspond to the k double stars $T_1, T_2, \ldots, T_k$, and where two vertices in the underlying graph are adjacent if the corresponding double stars are joined by an edge in T. Since T is a tree, so too is its underlying graph. Renaming the double stars $T_1, T_2, \ldots, T_k$ if necessary, we may assume that T_1 corresponds to a leaf in the underlying tree of T and that T_1 is joined to T_2 by an edge, say x_1x_2 where $x_i \in V(T_i)$ for $i \in [2]$. Further, renaming the vertices u_i and v_i if necessary, we may assume that v_ix_i is an edge of T, and so $u_1v_1x_1x_2v_2u_2$ is a path in T. We note that x_1 has degree 2 in T with v_1 and x_2 as its neighbors. Let $T' = T - V(T_1)$.

Then, $T' \in \mathcal{F}_{k',\Delta}$ where $k' = k - 1 \geq 2$. Applying the inductive hypothesis to T', we note that $st^-_{\gamma_t}(T') = 2\Delta - 1$.

Let S be a non-isolating set of vertices of T such that $\gamma_t(T - S) \leq 2k - 1$. Among all such sets, we choose S to have minimum cardinality. We show that $|S| \geq 2\Delta - 1$. Suppose, to the contrary, that $|S| \leq 2\Delta - 2$.

Let D be a minimum TD-set of $T - S$, and so $|D| = \gamma_t(T - S) \leq 2k - 1$. Let $D' = D \cap V(T')$ and $S' = S \cap V(T')$. Further, let $D_i = D \cap V(T_i)$ and $S_i = S \cap V(T_i)$ for $i \in [k]$.

Suppose that $D_1 = \emptyset$. In this case, S_1 contains all vertices of $V(T_1)$, except possibly for the vertex x_1, and so $|S| \geq |S_1| \geq 2\Delta - 1$, a contradiction. Hence, $|D_1| \geq 1$.

Suppose that $|D_1| = 1$. In this case, $D_1 = \{x_1\}$ and the set S_1 contains all vertices of $V(T_1) \setminus \{x_1\}$, except possibly for the vertex v_1. Thus, $|S_1| \geq 2\Delta - 2$. By assumption, $|S| = 2\Delta - 2$, implying that $|S| = |S_1| = 2\Delta - 2$ and that $S = V(T_1) \setminus \{x_1, v_1\}$. Thus, the graph $T - S$ is obtained from T' by adding the vertices x_1 and v_1 and adding the edges x_1x_2 and x_1v_1. Let u_2' be a leaf-neighbor of u_2 in the tree T_2. By rooting the tree T' at the vertex u_2, the set $\{u_2, u_2'\}$ can be extended to a maximum open packing of T' by adding to it two vertices from each tree T_i, where $i \in \{3, \ldots, k\}$, where from each tree T_i we add to the open packing the central vertex u_i or v_i of T_i at maximum distance from u_2, together with a leaf-neighbor of such a selected vertex in T_i. The resulting open packing of T' can be extended further to an open packing of $T - S$ by adding to it v_1 and x_1, implying that $\gamma_t(T - S) = \rho^0(T - S) \geq \rho^0(T') + 2 = 2(k - 1) + 2 = 2k$. This contradicts the fact that $\gamma_t(T - S) \leq 2k - 1$. Hence, $|D_1| \geq 2$.

Suppose that $x_1 \notin D_1$. In this case, S' is a non-isolating set of vertices of T'. Recall that $st^-_{\gamma_t}(T') = 2\Delta - 1$ and $|S'| \leq |S| < 2\Delta - 1$, implying that $\gamma_t(T' - S') \geq \gamma_t(T') = 2(k - 1)$. This in turn implies that $\gamma_t(T - S) = |D_1| + \gamma_t(T' - S') \geq 2 + 2(k - 1) = 2k$, a contradiction. Hence, $x_1 \in D_1$.

Suppose that $v_1 \notin D_1$. In this case, S_1 contains all $\Delta - 2$ leaf-neighbors of v_1. Moreover since $|D_1| \geq 2$, the set D_1 contains u_1 as well as a leaf-neighbor, say u_1', of u_1. We now consider the non-isolating set $S^* = S \setminus S_1$ of vertices of T. The set $(D \setminus \{u_1'\}) \cup \{v_1\}$ is a TD-set of $T - S^*$, and so $\gamma_t(T - S^*) \leq |D| = \gamma_t(T - S) \leq 2k - 1$. Thus, S^* is a non-isolating set of vertices of T such that $|S^*| < |S|$ and $\gamma_t(T - S^*) \leq 2k - 1$, contradicting our choice of the set S. Hence, $v_1 \in D_1$.

Suppose that $u_1 \in D_1$. In this case, $D_1 = \{u_1, v_1, x_1\}$. By the minimality of the set S, this implies that no leaf in T_1 belongs to S. Hence, $S = S'$ and $|S'| = |S| \leq 2\Delta - 2$. If v_2 is a vertex of $T - S$, then S' is a non-isolating set of vertices of T' and $D' \cup \{v_2\}$ is a TD-set of $T' - S'$, implying that $|D'| + 1 = |D' \cup \{v_2\}| \geq \gamma_t(T' - S') \geq \gamma_t(T') = 2(k - 1)$. Thus, $|D| = |D_1| + |D'| \geq 3 + (2k - 3) = 2k = \gamma_t(T)$, a contradiction. Therefore, $u_1 \notin D_1$, implying that $D_1 = \{v_1, x_1\}$. Further this implies that S_1 contains all $\Delta - 1$ leaf-neighbors of u_1, and so $|S_1| \geq \Delta - 1$.

This is true for each of the original double stars in T that corresponds to a leaf in the underlying tree of T. That is, if T_i is an arbitrary double star in T that corresponds to a leaf in the underlying tree of T for some $i \in [k]$, then $D_i = \{v_i, x_i\}$ and $|S_i| \geq \Delta - 1$. Since the underlying tree of T contains at least two leaves, this implies that $|S_i| \geq \Delta - 1$ for at least two different values of $i \in [k]$, say for i_1 and i_2. If $|D_i| \geq 2$ for all $i \in [k]$, then $|D| \geq 2k$, a contradiction. Hence, $|D_{i_3}| \leq 1$ for some $i_3 \in [k]$. If T_{i_3} corresponds to a leaf in the underlying tree of T, then as observed earlier, $|D_{i_3}| \geq 2$, a contradiction. Hence, T_{i_3} does not corresponds to a leaf in the underlying tree of T, and so i_3 is distinct from i_1 and i_2. Since $|D_{i_3}| \leq 1$, we note that neither u_{i_3} nor v_{i_3} belong to D_{i_3}, implying that at least one of u_{i_3} and v_{i_3} belong to S_{i_3}. Thus, $|S_{i_3}| \geq 1$. Thus, $|S| \geq |S_{i_1}| + |S_{i_2}| + |S_{i_3}| \geq 2(\Delta - 1) + 1 = 2\Delta - 1$, contradicting our supposition that $|S| \leq 2\Delta - 2$. We deduce, therefore, that $|S| \geq 2\Delta - 1$, implying that $st^-_{\gamma_t}(T) \geq 2\Delta - 1$.

Conversely, if we take $S = V(T_1)\backslash\{x_1\}$, then S is a non-isolating set of vertices of T such that $|S| = 2\Delta - 1$ and $\gamma_t(T - S) = 2k - 1$, and so $st^-_{\gamma_t}(T) \leq 2\Delta - 1$. Consequently, $st^-_{\gamma_t}(T) = 2\Delta - 1$. This completes the proof of Lemma 14. ∎

The γ_t^+-stability of a tree in the family $\mathcal{F}_\Delta$ is considerably less than its γ_t^--stability.

Lemma 15 *For $\Delta \geq 2$, if $T \in \mathcal{F}_\Delta$, then $st^+_{\gamma_t}(T) \leq \Delta - 1$.*

Proof. For $k \geq 2$ and $\Delta \geq 2$, let $T \in \mathcal{F}_{k,\Delta}$. Thus, T is the graph obtained from the disjoint union of k double stars $T_1, T_2, \ldots, T_k$, each isomorphic to $S(\Delta - 1, \Delta - 1)$, by adding $k - 1$ edges between leaves of these double stars so that the resulting graph is a tree with maximum degree Δ. We follow the notation introduced in the proof of Lemma 14. In particular, u_i and v_i denote the two central vertices of the double star T_i for $i \in [k]$. Recall that $\gamma_t(T) = 2k$. As in the proof of Lemma 14, we may assume that T_1 corresponds to a leaf in the underlying tree of T and that x_1x_2 is the edge joined T_1 and T_2, where $x_i \in V(T_i)$ for $i \in [2]$. Further, we may assume that v_1x_1 and x_2v_2 are edges of T.

Let S be the set of vertices consisting of v_1 and its $\Delta - 2$ leaf-neighbors in T. Thus, $S = N_T[v_1] \setminus \{u_1, x_1\}$. The tree $T - S$ has two components. Let T'_1 and T'_2 be the components of $T - S$ containing u_1 and x_1, respectively. We note that T'_1 is isomorphic to a star $K_{1,\Delta-1}$, and so $\gamma_t(T'_1) = 2$. Moreover, T'_2 is the tree obtained from T by removing all vertices in $V(T_1) \setminus \{x_1\}$. Let u'_2 be a leaf-neighbor of u_2 in the tree T_2. Analogously as shown in the proof of Lemma 14, the set $\{u_2, u'_2\}$ can be extended to a maximum open packing of T' by adding to it two vertices from each tree T_i, where $i \in \{3, \ldots, k\}$, where from each tree T_i we add to the open packing the central vertex u_i or v_i of T_i at maximum distance from u_2, together with a leaf-neighbor of such a selected vertex in T_i. The resulting open packing of T' can be extended further to an open packing of T'_2 by adding to it the vertex x_1, implying that $\gamma_t(T'_2) = \rho^0(T'_2) \geq 2k - 1$. Thus,

$\gamma_t(T-S) = \gamma_t(T'_1) + \gamma_t(T'_2) = 2k+1 > \gamma_t(T)$. Thus, S is a non-isolating set of vertices whose removal increases the total domination number, implying that $st^+_{\gamma_t}(T) \le |S| = \Delta - 1$. ■

We determine next the total domination stability of a tree in the family $\mathcal{H}_\Delta$.

Lemma 16 *For $\Delta \ge 2$, if $T \in \mathcal{H}_\Delta$, then $st_{\gamma_t}(T) = 2\Delta - 2$.*

Proof. If $\Delta = 2$, then the family $\mathcal{H}_\Delta$ is the family of all paths P_n, where $n \ge 7$ and $n \equiv 3 \pmod 4$. Thus in this case, by Proposition 10, we have $st_{\gamma_t}(T) = 2 = 2\Delta - 2$. Hence, in what follows we may assume that $\Delta \ge 3$, for otherwise the desired result follows. Thus, the tree $T \in \mathcal{H}_{k,\Delta}$ for some integer k where $\Delta \ge k \ge 2$ and is obtained from the disjoint union of k double stars $T_1, T_2, \ldots, T_k$, each isomorphic to $S(\Delta-1, \Delta-1)$, by selecting one leaf from each double star and identifying these k leaves into one new vertex, which we call w. Let u_i and v_i denote the two central vertices of the double star T_i for $i \in [k]$, where v_i is adjacent to w in T, and let $D = \cup_{i=1}^{k} \{u_i, v_i\}$. We note that the set D is the set of support vertices of T. Every TD-set in T contains all its support vertices, implying that $\gamma_t(T) \ge |D| = 2k$. However, the set of support vertices D is a TD-set of T, and so $\gamma_t(T) \le |D| = 2k$. Consequently, $\gamma_t(T) = 2k$, implying that the set D is the unique minimum TD-set in T.

We show firstly that $st_{\gamma_t}(T) \ge 2\Delta - 2$. Let S be a minimum non-isolating set of vertices of T such that $\gamma_t(T-S) \ne \gamma_t(T) = 2k$. Thus, $st_{\gamma_t}(T) = |S|$. Let R be a maximum open packing of the tree $T-S$. The set R contains at most one neighbor of u_i and at most one neighbor of v_i in T_i for each $i \in [k]$, implying that $|R \cap V(T_i)| \le 2$ for all $i \in [k]$. Thus if $w \notin R$, then $|R| \le 2k$. Suppose that $w \in R$. Since S is a minimum non-isolating set of vertices of T, the vertex w is not isolated in $T-S$. Renaming vertices if necessary, we may assume that v_1 is a neighbor of w in $T-S$. Since R is an open packing in $T-S$, the vertex w is the only neighbor of v_1 that belongs to R. As observed earlier, R contains at most one neighbor of u_1, and so $|R \cap V(T_1)| \le 1$, implying that once again that $|R| \le 2k$. Hence, $\rho^0(T-S) = |R| \le 2k$. Thus, by Theorem 5, $\gamma_t(T-S) = \rho^0(T-S) \le 2k$. Since $\gamma_t(T-S) \ne 2k$, this implies that $\gamma_t(T-S) < 2k$.

Since S is a non-isolating set of vertices of T, we note that if S contains a support vertex of T, then it contains all leaf-neighbors of that support vertex. By construction of the tree T, the support vertex u_i has $\Delta - 1$ leaf-neighbors, while the support vertex v_i has $\Delta - 2$ leaf-neighbors for all $i \in [k]$. Thus, if $|S \cap D| \ge 2$, then $|S| \ge 2 + 2(\Delta - 2) = 2\Delta - 2$. Hence, we may assume that $|S \cap D| \le 1$, for otherwise $st_{\gamma_t}(T) = |S| \ge 2\Delta - 2$, as claimed.

If $w \in S$, then since $|S \cap D| \le 1$, the forest $T-S$ contains k components, each of which requires exactly two vertices to totally dominate, and so $\gamma_t(T-S) = 2k$, a contradiction. Hence, $w \notin S$.

Let W be a minimum TD-set of $T-S$, and so $|W| = \gamma_t(T-S) \le 2k-1$. Suppose that $w \notin W$. By assumption, $|S \cap D| \le 1$, and so at least one of u_i and

v_i belong to $T - S$ for every $i \in [k]$. Thus, since $w \notin W$, the set W contains at least two vertices from $V(T_i)$ in order to totally dominate the vertices of $V(T_i)$ that belong to $T - S$, implying that $\gamma_t(T - S) = |W| \geq 2k$, a contradiction. Hence, $w \in W$. This in turn implies that at least two vertices of D do not belong to the set W. We note that all leaf-neighbors of the support vertices that do not belong to W, belong to the removed set S of vertices. Suppose that at least two support vertices at distance 2 from w do not belong to W. Renaming vertices if necessary, we may assume that u_1 and u_2 do not belong to W. Thus, the $2(\Delta - 1)$ leaf-neighbors of u_1 and u_2 belong to S, and so $|S| \geq 2\Delta - 2$, as claimed. Hence, we may assume, renaming vertices if necessary, that W contains all support vertex at distance 2 from w, except possibly for the vertex u_k. Thus, $u_i \in W$ for each $i \in [k-1]$. Further, in order to totally dominate the vertex u_i, the set W contains at least one neighbor of u_i for each $i \in [k-1]$. Hence, $|W \cap V(T_i)| \geq 2$ for all $i \in [k-1]$. Since $w \in W$ and $|W| \leq 2k-1$, this implies that $|W \cap V(T_i)| = 2$ for each $i \in [k-1]$ and that W contains no vertex from $V(T_k)$. This in turn implies that S contains all vertices of T_k, except possibly for the vertex v_k. Thus, $st_{\gamma_t}(T) = |S| \geq |V(T_k) \setminus \{v_k\}| = 2\Delta - 2$, as claimed.

Conversely, if we take $S = V(T_k) \setminus \{v_k\}$, then S is a non-isolating set of vertices of T such that $|S| = 2\Delta - 2$ and $\gamma_t(T - S) = 2k - 1$, and so $st_{\gamma_t}(T) \leq |S| = 2\Delta - 2$. Consequently, $st_{\gamma_t}(T) = 2\Delta - 2$. This completes the proof of Lemma 16. ■

5 Proof of Theorem 6

We are now in a position to prove Theorem 6. Recall its statement.

Theorem 6. *If T is a tree with maximum degree Δ satisfying $\gamma_t(T) \geq 3$, then the following hold:*
(a) $st^-_{\gamma_t}(T) \leq 2\Delta - 1$, with equality if and only if $T \in \mathcal{F}_\Delta$;
(b) $st_{\gamma_t}(T) \leq 2\Delta - 2$, and this bound is sharp for all $\Delta \geq 2$.

Proof. We first prove Part (a). The sufficiency follows from Lemma 14. To prove the necessity, let T be a tree with maximum degree Δ satisfying $\gamma_t(T) \geq 3$. Necessarily, $\Delta \geq 2$. If $\Delta = 2$, then $G \cong P_n$, where $n \geq 5$, and the result follows from Proposition 8 and Lemma 14, noting that in this case the family $\mathcal{F}_{k,\Delta}$ is the family of all paths P_{4k}, where $k \geq 2$. Hence we may assume that $\Delta \geq 3$, for otherwise the desired result follows.

Since $\gamma_t(T) \geq 3$, we note that $\mathrm{diam}(T) \geq 4$. Let u and r be two vertices at maximum distance apart in T. Necessarily, u and r are leaves and $d(u, v) = \mathrm{diam}(T)$. We now root the tree T at the vertex r. Let v be the parent of u, w the parent of v, x the parent of w, and y the parent of x. If $\mathrm{diam}(T) = 4$, then $y = r$; otherwise, $y \neq r$.

Claim 17 *If a child of w different from v is a support vertex, then $st^-_{\gamma_t}(T) \le \Delta$.*

Proof. Suppose that v' is a child of w different from v that is a support vertex. Let D be a minimum TD-set of T. Necessarily, $\{v, v', w\} \subseteq D$. We now consider the non-isolating set S of vertices of T consisting of v' and its children. Thus, $S = C(v') \cup \{v'\}$ and $|S| = d_T(v') \le \Delta$. Since $D \setminus \{v\}$ is a TD-set of $T - S$, we note that $\gamma_t(T-S) \le |D|-1 = \gamma_t(T)-1$, implying that $st^-_{\gamma_t}(T) \le |S| \le \Delta$. ($\Box$)

By Claim 17, we may assume that every child of w different from v is a leaf, for otherwise $st^-_{\gamma_t}(T) < 2\Delta - 1$.

Claim 18 *If v or w has degree less than Δ, then $st^-_{\gamma_t}(T) \le 2(\Delta - 1)$.*

Proof. Suppose that v or w has degree less than Δ. Thus, $d_T(v)+d_T(w) \le 2\Delta-1$. We now consider the non-isolating set S of vertices of T consisting of the vertex w and all descendants of w. Thus, S consists of v and w and their children; that is, $S = C(v) \cup C(w) \cup \{w\}$. By supposition, $|S| = d_T(v) + d_T(w) - 1 \le 2(\Delta - 1)$. Recall that y is the parent of x. Since $(D \setminus \{v, w\}) \cup \{y\}$ is a TD-set of $T - S$, we note that $\gamma_t(T-S) \le |D|-1 = \gamma_t(T)-1$, implying that $st^-_{\gamma_t}(T) \le |S| \le 2(\Delta-1)$. ($\Box$)

By Claim 18, we may assume that both v and w have degree Δ, for otherwise $st^-_{\gamma_t}(T) < 2\Delta - 1$.

Claim 19 *If there exists a minimum TD-set of T that contains a neighbor of x different from w or contains the vertex x, then $st^-_{\gamma_t}(T) \le 2(\Delta - 1)$.*

Proof. Let D be a minimum TD-set of T. Necessarily, $\{v, w\} \subset D$. Suppose that $x \in D$ or D contains a neighbor of x different from w. We now consider the non-isolating set S of vertices of T consisting of all descendants of w. Thus, $S = C(v) \cup C(w)$ and $|S| = d_T(v) + d_T(w) - 2 = 2(\Delta - 1)$. If $x \in D$, then let $D' = D \setminus \{v\}$. If $x \notin D$, then let $D' = (D \setminus \{v, w\}) \cup \{x\}$, noting that in this case D contains a neighbor of x different from w. In both cases, the set D' is a TD-set of $T - S$, and so $\gamma_t(T - S) \le |D'| = |D| - 1 = \gamma_t(T) - 1$, implying that $st^-_{\gamma_t}(T) \le |S| \le 2(\Delta - 1)$. ($\Box$)

By Claim 19, we may assume that no minimum TD-set of T contains the vertex x or contains a neighbor of x different from w, for otherwise $st^-_{\gamma_t}(T) < 2\Delta-1$. With this assumption, the degree of x is determined.

Claim 20 $d_T(x) = 2$.

Proof. Suppose that $d_T(x) \ge 3$. Let w' be a child of x different from w. If w' is a leaf, then x belongs to every minimum TD-set of T, contradicting our earlier assumption. If w' is not a leaf, then either w' is a support vertex or each child of w' is a support vertex. In both cases, the vertex w' can be chosen to belong to some minimum TD-set of T, contradicting our earlier assumption. ($\Box$)

We now consider the non-isolating set S of vertices of T consisting of all descendants of x. Thus, S consists of v and w and their children; that is, $S = C(v) \cup C(w) \cup \{w\}$ and $|S| = d_T(v) + d_T(w) - 1 = 2\Delta - 1$. Since $(D\setminus\{v, w\})\cup\{y\}$ is a TD-set of $T-S$, we note that $\gamma_t(T-S) \le |D|-1 = \gamma_t(T)-1$, implying that $st^-_{\gamma_t}(T) \le |S| = 2\Delta - 1$. This establishes the desired upper bound.

Suppose next that $st^-_{\gamma_t}(T) = 2\Delta - 1$. Our earlier Claims 17, 18, 19 and 20 imply that (i) every child of w different from v is a leaf, (ii) both v and w have degree Δ in T, and (iii) $d_T(x) = 2$, for otherwise $st^-_{\gamma_t}(T) < 2\Delta-1$, a contradiction.

We now interchange the roles of the vertices u and r in the tree T, and root the tree T at the vertex u. In the resulting rooted tree, let v_r denote the parent of r, w_r the parent of v_r, and x_r the parent of w_r. Analogously as before, we deduce that (i), (ii) and (iii) above hold where now v, w and x are replaced with v_r, w_r and x_r. If $x = x_r$, then the tree T is determined. However, in this case $\gamma_t(T) = 4$. Further, if S is the non-isolating set of vertices of T consisting of the $\Delta - 1$ leaf-neighbors of v together with the $\Delta - 1$ leaf-neighbors of v_r, then $\gamma_t(T - S) = |\{w, x, w_r\}| = 3 < \gamma_t(T)$, implying that $st^-_{\gamma_t}(T) < 2\Delta - 1$, a contradiction. Hence, $x \ne x_r$.

We now consider the tree T' obtained from T by removing x and all its descendants. The structure of the tree T implies that $\gamma_t(T') \ge 2$ since both v_r and w_r are support vertices in T'. If $\gamma_t(T') = 2$, then, since $x \ne x_r$, this implies that T' is isomorphic to a double star $S(\Delta - 1, \Delta - 1)$ with x_r as a leaf in T'. In this case, $T \in T_{2,\Delta}$, and so $T \in T_\Delta$, as desired. Hence, we may assume that $\gamma_t(T') \ge 3$. Since both v_r and w_r have degree Δ in T and their degrees remain unchanged in T', the tree T' is a tree with maximum degree Δ, implying that $st^-_{\gamma_t}(T') \le 2\Delta - 1$.

Every TD-set of T' can be extended to a TD-set of T by adding to it the vertices v and w, and so $\gamma_t(T) \le \gamma_t(T') + 2$. Conversely, let D be a minimum TD-set of T. Since D contains all support vertices of T, we note that $\{v, w\} \subset D$ and that D contains no leaf-neighbor of v or w. If $x \in D$, then we can replace x in D by the parent of y in the tree T rooted at r. Hence, the restriction of D to $V(T')$ is a TD-set of T', implying that $\gamma_t(T') \le |D|-2 = \gamma_t(T)-2$. Consequently, $\gamma_t(T) = \gamma_t(T') + 2$.

Suppose that $st^-_{\gamma_t}(T') < 2\Delta - 1$. Let S' be a minimum non-isolating set of vertices of T' that decreases the total domination number of T'. Thus, $|S'| < 2\Delta - 1$ and $\gamma_t(T' - S') < \gamma_t(T')$. Every minimum TD-set of $T' - S'$ can be extended to a TD-set of T by adding to it the vertices v and w, and so $\gamma_t(T-S') \le \gamma_t(T' - S') + 2 < \gamma_t(T') + 2 = \gamma_t(T)$. Hence, S' is a non-isolating set of vertices of T such that $\gamma_t(T - S') < \gamma_t(T')$, and so $st^-_{\gamma_t}(T) \le |S'| = st^-_{\gamma_t}(T') < 2\Delta - 1$, a contradiction. Therefore, $st^-_{\gamma_t}(T') = 2\Delta - 1$.

From the above observations, the tree T' has maximum degree Δ and satisfies $\gamma_t(T') \ge 3$ and $st^-_{\gamma_t}(T') = 2\Delta - 1$. Applying the inductive hypothesis to T', the tree $T' \in \mathcal{F}_\Delta$. Hence, T' is the graph obtained from the disjoint union of $k' \ge 2$ double stars $T_1, T_2, \dots, T_{k'}$, each isomorphic to $S(\Delta-1, \Delta-1)$, by adding $k'-1$ edges between leaves of these double stars so that the resulting graph is a tree with maximum degree Δ. We note that $\gamma_t(T') = 2k'$ and the $2k'$ central vertices of the k' double stars used in the construction of T' form a minimum TD-set of T'. Recall that xy is an edge of T and $y \in V(T')$.

Suppose that y is a central vertex of one of the k' double stars in the construction of T'. Let S be the non-isolating set of vertices of T consisting of v and all children of v and w. Thus, $|S| = 2\Delta - 2$ and the minimum TD-set of T' consisting of the $2k'$ central vertices of the k' double stars of T' can be extended to a TD-set of $T - S$ by adding to it the vertex x, and so $\gamma_t(T - S) \le \gamma_t(T') + 1 = \gamma_t(T) - 1$, implying that $st^-_{\gamma_t}(T) \le |S| = 2\Delta - 2$, a contradiction. Therefore, y is a leaf in one of the k' double stars in the construction of T'. By definition of the family $\mathcal{F}_\Delta$, the tree T therefore belongs to the family $\mathcal{F}_\Delta$. This completes the proof of Part (a).

If $T \in \mathcal{F}_\Delta$ for some $\Delta \ge 2$, then by Lemmas 14 and 15, $st_{\gamma_t}(T) = \min\{st^-_{\gamma_t}(T), st^+_{\gamma_t}(T)\} \le \min\{2\Delta - 1, \Delta - 1\} = \Delta - 1$. If $T \notin \mathcal{F}_\Delta$ for any $\Delta \ge 2$, then by Part (a), $st^-_{\gamma_t}(T) \le 2\Delta - 2$, implying that $st_{\gamma_t}(T) = \min\{st^-_{\gamma_t}(T), st^+_{\gamma_t}(T)\} \le 2\Delta - 2$. In both cases, $st_{\gamma_t}(T) \le 2\Delta - 2$. This establishes the upper bound in Part (b). The sharpness of this upper bound for all $\Delta \ge 2$ follows from Lemma 16. $\square$

6 Proof of Theorem 7

In this section, we prove Theorem 7. Our proof relies strongly on the results of Lemma 13 and Theorem 6. Recall the statement of Theorem 7.

Theorem 7. *If G is a connected graph with maximum degree Δ satisfying $\gamma_t(G) \ge 3$, then $st_{\gamma_t}(G) \le 2\Delta - 1$.*

Proof. Let G be a connected graph satisfying $\gamma_t(G) \ge 3$ and let $\Delta = \Delta(G)$. We note that $\Delta \ge 2$. If $\Delta = 2$ and G is a path, then $G \cong P_n$ where $n \ge 5$, and $st_{\gamma_t}(G) \le 2\Delta - 2$ by Proposition 10. If $\Delta = 2$ and G is a cycle, then $G \cong C_n$ where $n \ge 5$, and $st_{\gamma_t}(G) \le 2\Delta - 1$ by Proposition 12. Hence we may assume that $\Delta \ge 3$, for otherwise the desired result follows.

By Lemma 13, the graph G contains a spanning tree T such that $\gamma_t(T) = \gamma_t(G)$.

Let S be a non-isolating set of vertices in T of minimum size whose removal decreases the total domination number of T; that is, $\gamma_t(T - S) < \gamma_t(T)$. By Observation 4, we note that $|S| = st^-_{\gamma_t}(T) \le n - 2$. In particular, $T - S$ is not the null graph, implying that each component of $T - S$ has order at least 2.

Since adding edges cannot increase the total domination number, $\gamma_t(G - S) \le \gamma_t(T - S) < \gamma_t(T) = \gamma_t(G)$. Further, since each component of $T - S$ has order at least 2, the supergraph $G - S$ of $T - S$ is isolate-free and is not the null-graph. Thus, S is a non-isolating set of vertices in G whose removal decreases the total domination number of G, implying that $st^-_{\gamma_t}(G) \le |S| = st^-_{\gamma_t}(T)$.

By Theorem 6, $st^-_{\gamma_t}(T) \le 2\Delta(T) - 1$. From the observation that $\Delta(T) \le \Delta = \Delta(G)$, it follows that

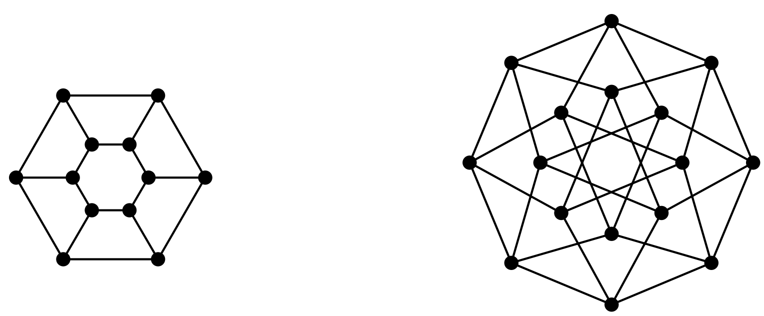
$$st_{\gamma_t}(G) = \min\{st^-_{\gamma_t}(G), st^+_{\gamma_t}(G)\} \le st^-_{\gamma_t}(G) \le st^-_{\gamma_t}(T) \le 2\Delta - 1.$$

This completes the proof of Theorem 7. ■

We discuss next graphs achieving the upper bound of Theorem 7; that is, graphs G with maximum degree Δ and $\gamma_t(G) \ge 3$ satisfying $st_{\gamma_t}(G) = 2\Delta - 1$.

By Propositions 10 and 12, if G is a connected graph with maximum degree $\Delta = 2$ satisfying $\gamma_t(T) \ge 3$, then $st_{\gamma_t}(G) \le 2\Delta - 1$, with equality if and only if $G \cong C_n$, where $n \ge 8$ and $n \equiv 0 \pmod 4$. This characterizes the extremal graphs of Theorem 7 in the case when $\Delta = 2$. For $\Delta \ge 3$, we have yet to characterize the extremal graphs of Theorem 7.

The upper bound of Theorem 7 cannot be improved when $\Delta = 3$ or $\Delta = 4$. For example, let G_3 be the 6-prism $C_6 \Box K_2$ which is depicted in Figure 1(a), and let G_4 be the 4-regular graph illustrated in Figure 1(b). The cubic graph G_3 satisfies $\Delta = 3$, $\gamma_t(G_3) = 4$ and $st_{\gamma_t}(G) = 5 = 2\Delta - 1$, while the 4-regular graph G_4 satisfies $\Delta = 4$, $\gamma_t(G_4) = 4$ and $st_{\gamma_t}(G_4) = 7 = 2\Delta - 1$.



(a) G_3 (b) G_4

Figure 1: The graphs G_3 and G_4

However we have yet to determine whether the upper bound in Theorem 7 is always achievable for large maximum degree $\Delta \ge 5$ and pose the following problem.

Problem 21 *For all fixed $\Delta \ge 2$, determine the smallest constant c_Δ such that every connected graph G with $\gamma_t(G) \ge 3$ and with maximum degree Δ satisfies $st_{\gamma_t}(G) \le c_\Delta \cdot \Delta$.*

As an immediate consequence of Theorem 7 and our above results, the following holds.

Corollary 22 *$c_2 = \frac{3}{2}$, $c_3 = \frac{5}{3}$, and $c_4 = \frac{7}{4}$.*

By Theorem 7, for every $\Delta \ge 2$, we note that $c_\Delta \le \frac{2\Delta-1}{\Delta}$. We close with the following question.

Question 23 *For every $\Delta \ge 2$, is it true that $c_\Delta = \frac{2\Delta-1}{\Delta}$?*

References

[1] D. Bauer, F. Harary, J. Nieminen and C. Suffel, Domination alternation sets in graphs. *Discrete Math.* **47** (1983), 153–161.

[2] R. C. Brigham, P. Z. Chinn and R. D. Dutton, Vertex domination-critical graphs. *Networks* **18** (1988), 173–179.

[3] R. C. Brigham, T. W. Haynes, M. A. Henning and D. F. Rall, Bicritical domination. *Discrete Math.* **305** (2005), 18–32.

[4] T. Burton and D. Sumner, Domination dot-critical graphs. *Discrete Math.* **306** (2006), 11–18.

[5] W. J. Desormeaux, T. W. Haynes and M. A. Henning, Total domination changing and stable graphs upon vertex removal. *Discrete Applied Math.* **159** (2011), 1548–1554.

[6] O. Favaron, D. Sumner and E. Wojcicka, The diameter of domination k-critical graphs. *J. Graph Theory* **18** (1994), 723–734.

[7] W. Goddard, T. W. Haynes, M. A. Henning and L. C. van der Merwe, The diameter of total domination vertex critical graphs. *Discrete Math.* **286** (2004), 255–261.

[8] T. W. Haynes, S. T. Hedetniemi, and P. J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, Inc. New York, 1998.

[9] T. W. Haynes, S. T. Hedetniemi, and P. J. Slater, *Domination in Graphs: Advanced Topics*, Marcel Dekker, Inc. New York, 1998.

[10] M. A. Henning and N. Jafari Rad, Total domination vertex critical graphs of high connectivity. *Discrete Applied Math.* **157** (2009), 1969–1973.

[11] M. A. Henning and A. Yeo, Perfect matchings in total domination critical graphs. *Graphs Combin.* **27** (2011), 685–701.

[12] M. A. Henning and A. Yeo, *Total domination in graphs (Springer Monographs in Mathematics)* 2013. ISBN: 978-1-4614-6524-9 (Print) 978-1-4614-6525-6 (Online).

[13] N. Jafari Rad, E. Sharifi, and M. Krzywkowski, Domination stability in graphs. *Discrete Math.* **339**(7) (2016), 1909–1914.

[14] D. F. Rall, Total domination in categorical products of graphs. *Discuss. Math. Graph Theory* **25** (2005), 35–44.

[15] D. Sumner and P. Blitch, Domination critical graphs. *J. Combin. Theory B* **34** (1983), 65–76.

[16] D. Sumner, Critical concept in domination. *Discrete Math.* **86** (1990), 33–46.

[17] H. Wang, L. Kang and E. Shan, Matching properties on total domination vertex critical graphs. *Graphs Combin.* **25**, 851—861.