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**A LOWER BOUND ON THE DOUBLE
OUTER-INDEPENDENT DOMINATION
NUMBER OF A TREE**

Abstract. A vertex of a graph is said to dominate itself and all of its neighbors. A double outer-independent dominating set of a graph G is a set D of vertices of G such that every vertex of G is dominated by at least two vertices of D , and the set $V(G) \setminus D$ is independent. The double outer-independent domination number of a graph G , denoted by $\gamma_d^{oi}(G)$, is the minimum cardinality of a double outer-independent dominating set of G . We prove that for every nontrivial tree T of order n , with l leaves and s support vertices we have $\gamma_d^{oi}(T) \geq (2n + l - s + 2)/3$, and we characterize the trees attaining this lower bound. We also give a constructive characterization of trees T such that $\gamma_d^{oi}(T) = (2n + 2)/3$.

1. Introduction

Let $G = (V, E)$ be a graph. By the neighborhood of a vertex v of G we mean the set $N_G(v) = \{u \in V(G) : uv \in E(G)\}$. The degree of a vertex v , denoted by $d_G(v)$, is the cardinality of its neighborhood. By a leaf we mean a vertex of degree one, while a support vertex is a vertex adjacent to a leaf. We say that a subset of $V(G)$ is independent if there is no edge between every two its vertices.

A vertex of a graph is said to dominate itself and all of its neighbors. A subset $D \subseteq V(G)$ is a dominating set of G if every vertex of G is dominated by at least one vertex of D , while it is a double dominating set of G if every vertex of G is dominated by at least two vertices of D . The domination (double domination, respectively) number of G , denoted by $\gamma(G)$ ($\gamma_d(G)$, respectively), is the minimum cardinality of a dominating (double dominating, respectively) set of G . Note that double domination is a type of k -tuple domination in which each vertex of a graph is dominated at least k times for a fixed positive integer k . The study of k -tuple domination was ini-

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tiated by Harary and Haynes [3]. For a comprehensive survey of domination in graphs, see [4, 5].

A subset $D \subseteq V(G)$ is a double outer-independent dominating set, abbreviated DOIDS, of G if every vertex of G is dominated by at least two vertices of D , and the set $V(G) \setminus D$ is independent. The double outer-independent domination number of a graph G , denoted by $\gamma_d^{oi}(G)$, is the minimum cardinality of a double outer-independent dominating set of G . A double outer-independent dominating set of G of minimum cardinality is called a $\gamma_d^{oi}(G)$ -set. Double outer-independent domination in graphs was introduced in [6].

Chellali and Haynes [2] proved the following lower bound on the total domination number of a tree. For every nontrivial tree T of order n with l leaves we have $\gamma_t(T) \geq (n - l + 2)/2$. They also characterized the extremal trees. Blidia, Chellali, and Favaron [1] established the following lower bound on the 2-domination number of a tree. For every nontrivial tree T of order n with l leaves we have $\gamma_2(T) \geq (n + l + 2)/3$. The extremal trees were also characterized.

We prove the following lower bound on the double outer-independent domination number of a tree. For every nontrivial tree T of order n , with l leaves and s support vertices we have $\gamma_d^{oi}(T) \geq (2n + l - s + 2)/3$. We also characterize the trees attaining this lower bound. We also give a constructive characterization of trees T such that $\gamma_d^{oi}(T) = (2n + 2)/3$.

2. Results

Since the one-vertex graph does not have double outer-independent dominating set, in this paper, by a tree we mean only a connected graph with no cycle, and which has at least two vertices.

We begin with the following two straightforward observations.

OBSERVATION 1. *Every leaf of a graph G is in every $\gamma_d^{oi}(G)$ -set.*

OBSERVATION 2. *Every support vertex of a graph G is in every $\gamma_d^{oi}(G)$ -set.*

We show that if T is a nontrivial tree of order n , with l leaves and s support vertices, then $\gamma_d^{oi}(T)$ is bounded below by $(2n + l - s + 2)/3$. For the purpose of characterizing the trees attaining this bound we introduce a family \mathcal{T} of trees $T = T_k$ that can be obtained as follows. Let T_1 be a path P_2 with vertices labeled x and y , and let $A(T_1) = \{x, y\}$. Let H be a path P_3 with leaves labeled u and z , and the support vertex labeled w . If k is a positive integer, then T_{k+1} can be obtained recursively from T_k by one of the following operations.

- Operation \mathcal{O}_1 : Attach a vertex v by joining it to any support vertex of T_k . Let $A(T_{k+1}) = A(T_k) \cup \{v\}$.
- Operation \mathcal{O}_2 : Attach a copy of H by joining u to a vertex of $A(T_k)$ which has degree at least two. Let $A(T_{k+1}) = A(T_k) \cup \{w, z\}$.
- Operation \mathcal{O}_3 : Attach a copy of H by joining u to a leaf of T_k which is the only one leaf among neighbors of its neighbor. Let $A(T_{k+1}) = A(T_k) \cup \{w, z\}$.

Now we prove that for every tree T of the family \mathcal{T} , the set $A(T)$ defined above is a DOIDS of minimum cardinality equal to $(2n + l - s + 2)/3$.

LEMMA 3. *If $T \in \mathcal{T}$, then the set $A(T)$ defined above is a $\gamma_d^{oi}(T)$ -set of size $(2n + l - s + 2)/3$.*

Proof. We use the terminology of the construction of the trees $T = T_k$, the set $A(T)$, and the graph H defined above. To show that $A(T)$ is a $\gamma_d^{oi}(T)$ -set of cardinality $(2n + l - s + 2)/3$ we use the induction on the number k of operations performed to construct T . If $T = T_1 = P_2$, then $(2n + l - s + 2)/3 = 2 = \gamma_d^{oi}(T)$. Let $k \geq 2$ be an integer. Assume that the result is true for every tree $T' = T_k$ of the family \mathcal{T} constructed by $k - 1$ operations. Let n' mean the order of the tree T' , l' the number of its leaves, and s' the number of support vertices. Let $T = T_{k+1}$ be a tree of the family \mathcal{T} constructed by k operations.

First assume that T is obtained from T' by operation \mathcal{O}_1 . We have $n = n' + 1$. It is easy to see that $A(T) = A(T') \cup \{v\}$ is DOIDS of the tree T . Of course, $\gamma_d^{oi}(T) = \gamma_d^{oi}(T') + 1$. If $T' = P_2$, then $l = l'$ and $s = s' - 1$. We get $\gamma_d^{oi}(T) = \gamma_d^{oi}(T') + 1 = (2n' + l' - s' + 2)/3 + 1 = (2n + l - s + 2)/3$. If $T' \neq P_2$, then $l = l' + 1$ and $s = s'$. Consequently, $\gamma_d^{oi}(T) = \gamma_d^{oi}(T') + 1 = (2n' + l' - s' + 2)/3 + 1 = (2n + l - s + 2)/3$.

Now assume that T is obtained from T' by operation \mathcal{O}_2 . We have $n = n' + 3$, $l = l' + 1$, and $s = s' + 1$. It is easy to see that $A(T) = A(T') \cup \{w, z\}$ is a DOIDS of the tree T . Let us observe that $\gamma_d^{oi}(T) = \gamma_d^{oi}(T') + 2$. Consequently, $\gamma_d^{oi}(T) = \gamma_d^{oi}(T') + 2 = (2n' + l' - s' + 2)/3 + 2 = (2n + l - s + 2)/3$.

Now assume that T is obtained from T' by operation \mathcal{O}_3 . We have $n = n' + 3$, $l = l'$, and $s = s'$. Similarly as when considering operation \mathcal{O}_2 we conclude that $A(T)$ is a DOIDS of the tree T and $\gamma_d^{oi}(T) = \gamma_d^{oi}(T') + 2$. Consequently, $\gamma_d^{oi}(T) = \gamma_d^{oi}(T') + 2 = (2n' + l' - s' + 2)/3 + 2 = (2n + l - s + 2)/3$. ■

Now we establish the main result, a lower bound on the double outer-independent domination number of a tree together with the characterization of the extremal trees.

THEOREM 4. *If T is a tree of order n , with l leaves and s support vertices, then $\gamma_d^{oi}(T) \geq (2n + l - s + 2)/3$ with equality if and only if $T \in \mathcal{T}$.*

Proof. If $\text{diam}(T) = 1$, then $T = P_2$. Thus $T \in \mathcal{T}$, and by Lemma 3 we have $\gamma_d^{oi}(T) = (2n + l - s + 2)/3$. Now assume that $\text{diam}(T) = 2$. Thus T is a star $K_{1,m}$. If $T = P_3$, then $T \in \mathcal{T}$ as it can be obtained from P_2 by operation \mathcal{O}_1 . If T is different than P_3 , then it is easy to see that T can be obtained from P_3 by a proper number of operations \mathcal{O}_1 . Thus every star T belongs to the family \mathcal{T} , and by Lemma 3 we have $\gamma_d^{oi}(T) = (2n + l - s + 2)/3$. Now assume that $\text{diam}(T) = 3$. Thus T is a double star. Observations 1 and 2 imply that every DOIDS of the tree T contains all leaves and all support vertices. Therefore the set $V(T)$ is the only one DOIDS of the tree T . This implies that $\gamma_d^{oi}(T) = n$. We have $l = n - 2$ and $s = 2$. Consequently, $(2n + l - s + 2)/3 = (2n + n - 2 - 2 + 2)/3 = (3n - 2)/3 = n - 2/3 < n = \gamma_d^{oi}(T)$, whence $T \notin \mathcal{T}$.

Now assume that $\text{diam}(T) \geq 4$. Thus the order of the tree T is an integer $n \geq 5$. If $T \in \mathcal{T}$, then by Lemma 3 we have $\gamma_d^{oi}(T) = (2n + l - s + 2)/3$. The result we obtain by the induction on the number n . Assume that the theorem is true for every tree T' of order $n' < n$, with l' leaves and s' support vertices.

First assume that some support vertex of T , say x , is adjacent to at least two leaves. One of them let us denote by y . Let $T' = T - y$. We have $n' = n - 1$, $l' = l - 1$, and $s' = s$. Of course, $\gamma_d^{oi}(T') = \gamma_d^{oi}(T) - 1$. Now we get $\gamma_d^{oi}(T) = \gamma_d^{oi}(T') + 1 \geq (2n' + l' - s' + 2)/3 + 1 = (2n - 2 + l - 1 - s + 2 + 3)/3 = (2n + l - s + 2)/3$. If $\gamma_d^{oi}(T) = (2n + l - s + 2)/3$, then obviously $\gamma_d^{oi}(T') = (2n' + l' - s' + 2)/3$. By the inductive hypothesis we have $T' \in \mathcal{T}$. The tree T can be obtained from T' by operation \mathcal{O}_1 . Thus $T \in \mathcal{T}$. Henceforth, we assume that every support vertex of T is adjacent to exactly one leaf.

We now root T at a vertex r of maximum eccentricity $\text{diam}(T)$. Let t be a leaf at maximum distance from r , v be the parent of t , u be the parent of v , and w be the parent of u in the rooted tree. By T_x let us denote the subtree induced by a vertex x and its descendants in the rooted tree T . We distinguish between the following two cases: $d_T(u) \geq 3$ and $d_T(u) = 2$.

Case 1. $d_T(u) \geq 3$. First assume that u has a child $b \neq v$ that is a support vertex. Let $T' = T - T_v$. We have $n' = n - 2$, $l' = l - 1$, and $s' = s - 1$. Let D be any $\gamma_d^{oi}(T)$ -set. By Observations 1 and 2 we have $t, v, b \in D$. If $u \in D$, then it is easy to observe that $D \setminus \{v, t\}$ is a DOIDS of the tree T' . Now assume that $u \notin D$. We have $d_T(u) \geq 3$, thus $d_{T'}(u) \geq 2$. Since $V(T) \setminus D$ is independent, every neighbor of u belongs to the set D .

Let us observe that $D \setminus \{v, t\}$ is a DOIDS of the tree T' as u has at least two neighbors in $D \setminus \{v, t\}$. Now we conclude that $\gamma_d^{oi}(T') \leq \gamma_d^{oi}(T) - 2$. We get $\gamma_d^{oi}(T) \geq \gamma_d^{oi}(T') + 2 \geq (2n' + l' - s' + 2)/3 + 2 = (2n - 4 + l - 1 - s + 1 + 2 + 6)/3 = (2n + l - s + 4)/3 > (2n + l - s + 2)/3$.

Now assume that v is the only one support vertex among the descendants of u . Thus u is a parent of a leaf, say x . Let $T' = T - T_x$. We have $n' = n - 1$, $l' = l - 1$, and $s' = s - 1$. Let D be a $\gamma_d^{oi}(T)$ -set. By Observations 1 and 2 we have $x, u, v \in D$. It is easy to observe that $D \setminus \{x\}$ is a DOIDS of the tree T' . This implies that $\gamma_d^{oi}(T') \leq \gamma_d^{oi}(T) - 1$. Now we get $\gamma_d^{oi}(T) \geq \gamma_d^{oi}(T') + 1 \geq (2n' + l' - s' + 2)/3 + 1 = (2n - 2 + l - 1 - s + 1 + 2 + 3)/3 = (2n + l - s + 3)/3 > (2n + l - s + 2)/3$, whence $T \notin \mathcal{T}$.

Case 2. $d_T(u) = 2$. The parent of w let us denote by d . Let D be any $\gamma_d^{oi}(T)$ -set. By Observations 1 and 2 we have $t, v \in D$. If $u \notin D$, then $w \in D$ as $V(T) \setminus D$ is independent. Let $T' = T - T_u$. We have $n' = n - 3$. It is easy to see that $D \setminus \{v, t\}$ is a DOIDS of the tree T' . Now assume that $u \in D$. If $w \in D$, then no neighbor of w besides u belongs to the set D , otherwise $D \setminus \{u\}$ is a DOIDS of the tree T , a contradiction to the minimality of D . It is easy to observe that $D \cup \{d\} \setminus \{u, v, t\}$ is a DOIDS of the tree T' . If $w \notin D$, then it is easy to see that $D \cup \{w\} \setminus \{u, v, t\}$ is a DOIDS of the tree T' . Now we conclude that $\gamma_d^{oi}(T') \leq \gamma_d^{oi}(T) - 2$. We consider the following two possibilities: $d_T(w) = 2$ and $d_T(w) \geq 3$.

First assume that $d_T(w) = 2$. We have $l' = l$. If d is adjacent to a leaf in T , then $s' = s - 1$. Consequently, $\gamma_d^{oi}(T) \geq \gamma_d^{oi}(T') + 2 \geq (2n' + l' - s' + 2)/3 + 2 = (2n - 6 + l - s + 1 + 2 + 6)/3 = (2n + l - s + 3)/3 > (2n + l - s + 2)/3$. Now assume that d is not adjacent to any leaf in T . Thus $s' = s$. Now we get $\gamma_d^{oi}(T) \geq \gamma_d^{oi}(T') + 2 \geq (2n' + l' - s' + 2)/3 + 2 = (2n - 6 + l - s + 2 + 6)/3 = (2n + l - s + 2)/3$. If $\gamma_d^{oi}(T) = (2n + l - s + 2)/3$, then obviously $\gamma_d^{oi}(T') = (2n' + l' - s' + 2)/3$. By the inductive hypothesis we have $T' \in \mathcal{T}$. The tree T can be obtained from T' by operation \mathcal{O}_3 . Thus $T \in \mathcal{T}$.

Now assume that $d_T(w) \geq 3$. We have $l' = l - 1$ and $s' = s - 1$. Now we get $\gamma_d^{oi}(T) \geq \gamma_d^{oi}(T') + 2 \geq (2n' + l' - s' + 2)/3 + 2 = (2n - 6 + l - 1 - s + 1 + 2 + 6)/3 = (2n + l - s + 2)/3$. If $\gamma_d^{oi}(T) = (2n + l - s + 2)/3$, then obviously $\gamma_d^{oi}(T') = (2n' + l' - s' + 2)/3$. By the inductive hypothesis we have $T' \in \mathcal{T}$. The tree T can be obtained from T' by operation \mathcal{O}_3 . Thus $T \in \mathcal{T}$. ■

Since the number of leaves of a tree is greater than or equal to the number of its support vertices, we get the following corollary.

COROLLARY 5. *For every tree T we have $\gamma_d^{oi}(T) \geq (2n + 2)/3$.*

Now we characterize the trees attaining this bound. For this purpose we introduce a family \mathcal{F} of trees $T = T_k$ that can be obtained as follows. Let T_1 be a path P_2 with vertices labeled x and y , and let $B(T_1) = \{x, y\}$. Let H be a path P_3 with leaves labeled u and z , and the support vertex labeled w . If k is a positive integer, then T_{k+1} can be obtained recursively from T_k by one of the following operations.

- Operation \mathcal{X}_1 : Attach a copy of H by joining u to a vertex of $B(T_k)$ which has degree at least two. Let $B(T_{k+1}) = B(T_k) \cup \{w, z\}$.
- Operation \mathcal{X}_2 : Attach a copy of H by joining u to a leaf of T_k which is the only one leaf among neighbors of its neighbor. Let $B(T_{k+1}) = B(T_k) \cup \{w, z\}$.

Now we prove that for every tree T of the family \mathcal{F} , the set $B(T)$ defined above is a DOIDS of minimum cardinality equal to $(2n + 2)/3$.

LEMMA 6. *If $T \in \mathcal{F}$, then the set $B(T)$ defined above is a $\gamma_d^{oi}(T)$ -set of size $(2n + 2)/3$.*

Proof. The definitions of the families \mathcal{T} and \mathcal{F} imply that $\mathcal{F} \subseteq \mathcal{T}$. Thus $T \in \mathcal{T}$. By Lemma 3, the set $A(T) = B(T)$ is a $\gamma_d^{oi}(T)$ -set of size $(2n + l - s + 2)/3$. Obviously, for $T_1 = P_2$ we have $l = s$. Let us observe that performing neither the operation \mathcal{X}_1 nor the operation \mathcal{X}_2 disturbs the equality $l = s$. Therefore $l = s$, and consequently, $(2n + l - s + 2)/3 = (2n + 2)/3$. ■

Now we prove a lower bound on the double outer-independent domination number of a tree in terms of the number of vertices, together with the characterization of the extremal trees.

THEOREM 7. *If T is a tree of order n , then $\gamma_d^{oi}(T) \geq (2n + 2)/3$ with equality if and only if $T \in \mathcal{F}$.*

Proof. The bound is true by Corollary 5. If $T \in \mathcal{F}$, then by Lemma 6 we have $\gamma_d^{oi}(T) = (2n + 2)/3$. Now assume that for a tree T we have $\gamma_d^{oi}(T) = (2n + 2)/3$. The number of leaves of every tree is greater than or equal to the number of its support vertices, thus $l \geq s$. By Theorem 4 we have $\gamma_d^{oi}(T) \geq (2n + l - s + 2)/3$. This implies that $l = s$. We have $\gamma_d^{oi}(T) = (2n + 2)/3 = (2n + l - s + 2)/3$. By Theorem 4 we have $T \in \mathcal{T}$. Suppose that T is obtained from $T_1 = P_2$ in a way such that the operation \mathcal{O}_1 is used at least once. Let us observe that $l > s$ as $l(P_2) = s(P_2)$, the operation \mathcal{O}_1 increases l not changing s , and the operations \mathcal{O}_2 and \mathcal{O}_3 do not disturb the equality $l = s$. This is a contradiction to that $l = s$. Therefore the operation \mathcal{O}_1 was not used to obtain the tree T . Since the operations \mathcal{O}_2 and \mathcal{O}_3 are identical to operations \mathcal{X}_1 and \mathcal{X}_2 , respectively, we conclude that $T \in \mathcal{F}$. ■

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