Minimal double dominating sets in trees

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Abstract. We provide an algorithm for listing all minimal double dominating sets of a tree of order n in time $\mathcal{O}(1.3248^n)$. This implies that every tree has at most 1.3248^n minimal double dominating sets. We also show that this bound is tight.

Keywords: domination, double domination, minimal double dominating set, tree, combinatorial bound, exact exponential algorithm, listing algorithm

1 Introduction

Let $G = (V, E)$ be a graph. The order of a graph is the number of its vertices. By the neighborhood of a vertex v of G we mean the set $N_G(v) = \{u \in V(G) : uv \in E(G)\}$. The degree of a vertex v, denoted by $d_G(v)$, is the cardinality of its neighborhood. By a leaf we mean a vertex of degree one, while a support vertex is a vertex adjacent to a leaf. We say that a support vertex is strong (weak, respectively) if it is adjacent to at least two leaves (exactly one leaf, respectively). The distance between two vertices of a graph is the number of edges in a shortest path connecting them. The eccentricity of a vertex is the greatest distance between it and any other vertex. The diameter of a graph G, denoted by $\mathrm{diam}(G)$, is the maximum eccentricity among all vertices of G. A path on n vertices we denote by P_n.

A vertex of a graph is said to dominate itself and all of its neighbors. A subset $D \subseteq V(G)$ is a dominating set of G if every vertex of G is dominated by at least one vertex of D, while it is a double dominating set of G if every vertex of G is dominated by at least two vertices of D. A dominating (double dominating, respectively) set D is minimal if no proper subset of D is a dominating (double dominating, respectively) set of G. A minimal double dominating set is abbreviated as mdds. Double domination in graphs was introduced by Harary and Haynes [6]. For a comprehensive survey of domination in graphs, see [7, 8].

Observation 1 *Every leaf of a graph G is in every DDS of G.*

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Observation 2 *Every support vertex of a graph G is in every DDS of G.*

One of the typical questions in graph theory is how many subgraphs of a given property a graph on n vertices can have. For example, the famous Moon and Moser theorem [12] says that every graph on n vertices has at most $3^{n/3}$ maximal independent sets.

Combinatorial bounds are of interest not only on their own, but also because they are used for algorithm design as well. Lawler [11] used the Moon-Moser bound on the number of maximal independent sets to construct an $(1+\sqrt[3]{3})^n \cdot n^{\mathcal{O}(1)}$ time graph coloring algorithm, which was the fastest one known for twenty-five years. For an overview of the field, see [5].

Fomin et al. [4] constructed an algorithm for listing all minimal dominating sets of a graph on n vertices in time $\mathcal{O}(1.7159^n)$. They also presented graphs ($n/6$ disjoint copies of the octahedron) having $15^{n/6} \approx 1.5704^n$ minimal dominating sets. This establishes a lower bound on the running time of an algorithm for listing all minimal dominating sets of a given graph.

The number of maximal independent sets in trees was investigated in [13]. Couturier et al. [3] considered minimal dominating sets in various classes of graphs. The authors of [9] investigated the enumeration of minimal dominating sets in graphs.

Bród and Skupień [1] gave bounds on the number of dominating sets of a tree. They also characterized the extremal trees. The authors of [2] investigated the number of minimal dominating sets in trees containing all leaves.

In [10] an algorithm was given for listing all minimal dominating sets of a tree of order n in time $\mathcal{O}(1.4656^n)$. This implies that every tree has at most 1.4656^n minimal dominating sets. An infinite family of trees for which the number of minimal dominating sets exceeds 1.4167^n was also given. This establishes a lower bound on the running time of an algorithm for listing all minimal dominating sets of a given tree.

We provide an algorithm for listing all minimal double dominating sets of a tree of order n in time $\mathcal{O}(1.3248^n)$. This implies that every tree has at most 1.3248^n minimal double dominating sets. We also show that this bound is tight.

2 Results

We describe a recursive algorithm which lists all minimal double dominating sets of a given input tree. We prove that the running time of this algorithm is $\mathcal{O}(1.3248^n)$, implying that every tree has at most 1.3248^n minimal double dominating sets.

Theorem 3 *Every tree T of order n has at most α^n minimal double dominating sets, where $\alpha \approx 1.32472$ is the positive solution of the equation $x^3 - x - 1 = 0$, and all those sets can be listed in time $\mathcal{O}(1.3248^n)$.*

Proof. The family of sets returned by our algorithm is denoted by $\mathcal{F}(T)$. To obtain the upper bound on the number of minimal double dominating sets of a tree, we prove that the algorithm lists these sets in time $\mathcal{O}(1.3248^n)$. If $\mathrm{diam}(T) \le 3$, then let $\mathcal{F}(T) = \{V(T)\}$. Every vertex of T is a leaf or a support vertex. Observations 1 and 2 imply that $V(T)$ is the only mdds of T. We have $n \ge 2$ and $|\mathcal{F}(T)| = 1$. Obviously, $1 < \alpha^n$.

Now assume that $\mathrm{diam}(T) \ge 4$. Thus the order n of the tree T is at least five. The results we obtain by the induction on the number n. Assume that they are true for every tree T' of order $n' < n$.

First assume that some support vertex of T, say x, is strong. Let y and z be leaves adjacent to x. Let $T' = T - y$, and let

$$\mathcal{F}(T) = \{D' \cup \{y\} \colon D' \in \mathcal{F}(T')\}.$$

Let D' be an mdds of the tree T'. By Observation 2 we have $x \in D'$. It is easy to see that $D' \cup \{y\}$ is an mdds of T. Thus all elements of the family $\mathcal{F}(T)$ are minimal double dominating sets of the tree T. Now let D be any mdds of the tree T. By Observations 1 and 2 we have $x, y, z \in D$. Let us observe that $D \setminus \{y\}$ is an mdds of the tree T' as the vertex x is still dominated at least twice. By the inductive hypothesis we have $D \setminus \{y\} \in \mathcal{F}(T')$. Therefore the family $\mathcal{F}(T)$ contains all minimal double dominating sets of the tree T. We now get $|\mathcal{F}(T)| = |\mathcal{F}(T')| \le \alpha^{n-1} < \alpha^n$. Henceforth, we can assume that every support vertex of T is weak.

We now root T at a vertex r of maximum eccentricity $\mathrm{diam}(T)$. Let t be a leaf at maximum distance from r, v be the parent of t, u be the parent of v, and w be the parent of u in the rooted tree. If $\mathrm{diam}(T) \ge 5$, then let d be the parent of w. By T_x we denote the subtree induced by a vertex x and its descendants in the rooted tree T.

Assume that u is adjacent to a leaf, say x. Let $T' = T - T_v$, and let

$$\mathcal{F}(T) = \{D' \cup \{v, t\} \colon D' \in \mathcal{F}(T')\}.$$

Let us observe that all elements of the family $\mathcal{F}(T)$ are minimal double dominating sets of the tree T. Now let D be any mdds of the tree T. By Observations 1 and 2 we have $t, x, v, u \in D$. It is easy to observe that $D \setminus \{v, t\}$ is an mdds of the tree T'. By the inductive hypothesis we have $D \setminus \{v, t\} \in \mathcal{F}(T')$. Therefore the family $\mathcal{F}(T)$ contains all minimal double dominating sets of the tree T. We now get $|\mathcal{F}(T)| = |\mathcal{F}(T')| \le \alpha^{n-2} < \alpha^n$.

Now assume that all children of u are support vertices. Assume that $d_T(u) \ge 4$. Let $T' = T - T_v$, and let

$$\mathcal{F}(T) = \{D' \cup \{v, t\} \colon D' \in \mathcal{F}(T')\}.$$

Let us observe that all elements of the family $\mathcal{F}(T)$ are minimal double dominating sets of the tree T. Now let D be any mdds of the tree T. By Observations 1 and 2 we have $v, t \in D$. Let us observe that $D \setminus \{v, t\}$ is an mdds of the tree T' as the vertex u is still dominated at least twice. By the inductive hypothesis we have $D \setminus \{v, t\} \in \mathcal{F}(T')$. Therefore the family $\mathcal{F}(T)$ contains all minimal double dominating sets of the tree T. We now get $|\mathcal{F}(T)| = |\mathcal{F}(T')| \le \alpha^{n-2} < \alpha^n$.

Now assume that $d_T(u) = 3$. Let x be the child of u other than v. The leaf adjacent to x we denote by y. Let $T' = T - T_u$ and $T'' = T - T_v - y$. Let $\mathcal{F}(T)$ be a family as follows,

$$\{D' \cup \{t, v, x, y\} \colon D' \in \mathcal{F}(T')\}$$
$$\cup \{D'' \cup \{v, t, y\} \colon D'' \in \mathcal{F}(T'') \text{ and } D'' \setminus \{u, x\} \notin \mathcal{F}(T')\}.$$

Let us observe that all elements of the family $\mathcal{F}(T)$ are minimal double dominating sets of the tree T. Now let D be any mdds of the tree T. By Observations 1 and 2 we have $v, t, x, y \in D$. If $u \notin D$, then observe that $D \setminus \{v, t, x, y\}$ is an mdds of the tree T'. By the inductive hypothesis we have $D \setminus \{v, t, x, y\} \in \mathcal{F}(T')$. Now assume that $u \in D$. It is easy to observe that $D \setminus \{v, t, y\}$ is an mdds of the tree T''. By the inductive hypothesis we have $D \setminus \{v, t, y\} \in \mathcal{F}(T'')$. Let us observe that $D \setminus \{u, v, t, x, y\}$ is not a double dominating set of the tree T', otherwise $D \setminus \{u\}$ is a double dominating set of the tree T, a contradiction to the minimality of D. Therefore the family $\mathcal{F}(T)$ contains all minimal double dominating sets of the tree T. We now get $|\mathcal{F}(T)| = |\mathcal{F}(T')| + |\{D'' \in \mathcal{F}(T'') \colon D'' \setminus \{u, x\} \notin \mathcal{F}(T')\}| \le |\mathcal{F}(T')| + |\mathcal{F}(T'')| \le \alpha^{n-5} + \alpha^{n-3} = \alpha^{n-5}(\alpha^2 + 1) < \alpha^{n-5} \cdot \alpha^5 = \alpha^n$.

Now assume that $d_T(u) = 2$. Assume that $d_T(w) \ge 3$. First assume that w is adjacent to a leaf, say k. Let $T' = T - T_u$, and let

$$\mathcal{F}(T) = \{D' \cup \{v, t\} \colon D' \in \mathcal{F}(T')\}.$$

Let us observe that all elements of the family $\mathcal{F}(T)$ are minimal double dominating sets of the tree T. Now let D be any mdds of the tree T. By Observations 1 and 2 we have $v, t, w, k \in D$. We have $u \notin D$ as the set D is minimal. Observe that $D \setminus \{v, t\}$ is an mdds of the tree T'. By the inductive hypothesis we have $D \setminus \{v, t\} \in \mathcal{F}(T')$. Therefore the family $\mathcal{F}(T)$ contains all minimal double dominating sets of the tree T. We now get $|\mathcal{F}(T)| = |\mathcal{F}(T')| \le \alpha^{n-3} < \alpha^n$.

Now assume that there is a child of w, say k, such that the distance of w to the most distant vertex of T_k is two. Thus k is a support vertex of degree two. The leaf adjacent to k we denote by l. Let $T' = T - T_u - l$ and $T'' = T - T_w$. Let

$$\mathcal{F}(T) = \{D' \cup \{v, t, l\} \colon D' \in \mathcal{F}(T')\} \cup \{D'' \cup V(T_w) \setminus \{w\} \colon D'' \in \mathcal{F}(T'')\}.$$

Let us observe that all elements of the family $\mathcal{F}(T)$ are minimal double dominating sets of the tree T. Now let D be any mdds of the tree T. By Observations 1 and 2 we have $v, t, k, l \in D$. If $u \notin D$, then $w \in D$ as the vertex u has to be dominated twice. It is easy to observe that $D \setminus \{v, t, l\}$ is an mdds of the tree T'. By the inductive hypothesis we have $D \setminus \{v, t, l\} \in \mathcal{F}(T')$. Now assume that $u \in D$. We have $w \notin D$, otherwise $D \setminus \{u\}$ is a double dominating set of the tree T, a contradiction to the minimality of D. Observe that $D \cap V(T'')$ is an mdds of the tree T''. By the inductive hypothesis we have $D \cap V(T'') \in \mathcal{F}(T'')$. Therefore the family $\mathcal{F}(T)$ contains all minimal double dominating sets of the tree T. We now get $|\mathcal{F}(T)| = |\mathcal{F}(T')| + |\mathcal{F}(T'')| \le \alpha^{n-4} + \alpha^{n-6} = \alpha^{n-6}(\alpha^2 + 1) < \alpha^{n-6} \cdot \alpha^6 = \alpha^n$.

Now assume that for every child of w, say k, the distance of w to the most distant vertex of T_k is three. Due to the earlier analysis of the degree of the vertex u, which is a child of w, it suffices to consider only the possibility when T_k is a path P_3. Let $T' = T - T_w$. Let T'' (T''', respectively) be a tree that differs from T' only in that it has the vertex w (the vertices w and u, respectively). Let $\mathcal{F}(T)$ be a family as follows,

$$\begin{aligned} &\{D' \cup V(T_w) \setminus \{w\} \colon D' \in \mathcal{F}(T')\} \\ \cup\, &\{D'' \cup V(T_w) \setminus (N_T(w) \setminus \{d\}) \colon D'' \in \mathcal{F}(T'')\} \\ \cup\, &\{D''' \cup V(T_w) \setminus (N_T(w) \setminus \{x\}) \colon d \notin D''' \in \mathcal{F}(T''') \text{ and } x \in N_T(w) \setminus \{d\}\}. \end{aligned}$$

Let us observe that all elements of the family $\mathcal{F}(T)$ are minimal double dominating sets of the tree T. Now let D be any mdds of the tree T. If $w \notin D$, then observe that $D \cap V(T')$ is an mdds of the tree T'. By the inductive hypothesis we have $D \cap V(T') \in \mathcal{F}(T')$. Now assume that $w \in D$. If no child of w belongs to the set D, then observe that $D \cap V(T'')$ is an mdds of the tree T''. By the inductive hypothesis we have $D \cap V(T'') \in \mathcal{F}(T'')$. Now assume that some child of w, say x, belongs to the set D. Let us observe that $(D \cup \{u\}) \cap V(T''')$ is an mdds of the tree T'''. By the inductive hypothesis we have $(D \cup \{u\}) \cap V(T''') \in \mathcal{F}(T''')$. Therefore the family $\mathcal{F}(T)$ contains all minimal double dominating sets of the tree T. We now get $|\mathcal{F}(T)| = |\mathcal{F}(T')| + |\mathcal{F}(T'')| + (d_T(w) - 1) \cdot |\{D''' \in \mathcal{F}(T''') \colon d \notin D'''\}|$ $\leq |\mathcal{F}(T')| + |\mathcal{F}(T'')| + (d_T(w) - 1) \cdot |\mathcal{F}(T''')| \leq \alpha^{n-3d_T(w)+2} + \alpha^{n-3d_T(w)+3}$ $+(d_T(w) - 1) \cdot \alpha^{n-3d_T(w)+4}$. To show that $\alpha^{n-3d_T(w)+2} + \alpha^{n-3d_T(w)+3} + (d_T(w)$ $-1) \cdot \alpha^{n-3d_T(w)+4} < \alpha^n$, it suffices to show that $\alpha^2 + \alpha^3 + (d_T(w) - 1) \cdot \alpha^4$ $< \alpha^{3d_T(w)}$. We prove this by the induction on the degree of the vertex w. For $d_T(w) = 3$ we have $\alpha^2 + \alpha^3 + (d_T(w) - 1) \cdot \alpha^4 = 2\alpha^4 + \alpha^3 + \alpha^2 = 2\alpha^4 + \alpha^2(\alpha + 1)$ $= 2\alpha^4 + \alpha^5 = \alpha^4(\alpha + 1) + \alpha^4 = \alpha^7 + \alpha^4 = \alpha^6(\alpha^3 - 1) + \alpha^4 = \alpha^9 + \alpha^4 - \alpha^6$ $< \alpha^9 = \alpha^{3d_T(w)}$. We now prove that if the inequality $\alpha^2 + \alpha^3 + (k-1) \cdot \alpha^4 < \alpha^{3k}$ is satisfied for an integer $k = d_T(w) \geq 3$, then it is also satisfied for $k+1$. We have $\alpha^2 + \alpha^3 + k\alpha^4 = \alpha^2 + \alpha^3 + (k-1) \cdot \alpha^4 + \alpha^4 < \alpha^{3k} + \alpha^4 < \alpha^{3k} + \alpha^{3k+1} = \alpha^{3k+3}$.

Now assume that $d_T(w) = 2$. If $d_T(d) = 1$, then let $\mathcal{F}(T) = \{\{d, w, v, t\}\}$. The tree T is a path P_5. It is easy to observe that $\{d, w, v, t\}$ is the only mdds of the tree T. We have $n = 5$ and $|\mathcal{F}(T)| = 1$. Obviously, $1 < \alpha^5$. Now assume that $d_T(d) \geq 2$. Due to the earlier analysis of the degrees of the vertices w and u, we may assume that for every child of d, say k, the tree T_k is a path on at most four vertices. Let $T' = T - T_u$, $T'' = T - T_w$ and $T''' = T - T_d$. If T''' is a single vertex, then let $\mathcal{F}(T) = \{\{r, d, w, v, t\}, \{r, d, u, v, t\}\}$. The tree T is a path P_6. Let us observe that $\{r, d, w, v, t\}$ and $\{r, d, u, v, t\}$ are the only two minimal double dominating sets of the tree T. We have $n = 6$ and $|\mathcal{F}(T)| = 2$. Obviously, $2 < \alpha^6$. Now assume that $|V(T''')| \geq 2$. Let $\mathcal{F}(T)$ be a family as follows,

$$\begin{aligned} &\{D' \cup \{v, t\} \colon D' \in \mathcal{F}(T')\} \\ \cup\, &\{D'' \cup \{u, v, t\} \colon d \in D'' \in \mathcal{F}(T'')\} \\ \cup\, &\{D''' \cup V(T_d) \setminus \{d\} \colon D''' \in \mathcal{F}(T''')\}, \end{aligned}$$

where the third component is ignored if d is adjacent to a leaf. Let us observe that all elements of the family $\mathcal{F}(T)$ are minimal double dominating sets of the tree T. Now let D be any mdds of the tree T. By Observations 1 and 2 we have $v, t \in D$. If $u \notin D$, then observe that $D \setminus \{v, t\}$ is an mdds of the tree T'. By the inductive hypothesis we have $D \setminus \{v, t\} \in \mathcal{F}(T')$. Now assume that $u \in D$. If $w \notin D$, then observe that $D \setminus \{u, v, t\}$ is an mdds of the tree T''. By the inductive hypothesis we have $D \setminus \{u, v, t\} \in \mathcal{F}(T'')$. Now assume that $w \in D$. We have $d \notin D$, otherwise $D \setminus \{u\}$ is a double dominating set of the tree T, a contradiction to the minimality of D. Observe that $D \cap V(T''')$ is an mdds of the tree T'''. By the inductive hypothesis we have $D \cap V(T''') \in \mathcal{F}(T''')$. Therefore the family $\mathcal{F}(T)$ contains all minimal double dominating sets of the tree T. We now get $|\mathcal{F}(T)| = |\mathcal{F}(T')| + |\{D'' \in \mathcal{F}(T'')\colon d \in D''\}| + |\mathcal{F}(T''')|$ $\leq |\mathcal{F}(T')| + |\mathcal{F}(T'')| + |\mathcal{F}(T''')| \leq \alpha^{n-3} + \alpha^{n-4} + \alpha^{n-5} = \alpha^{n-5}(\alpha^2 + \alpha + 1)$ $= \alpha^{n-5}(\alpha^2 + \alpha^3) = \alpha^{n-3}(\alpha + 1) = \alpha^{n-3} \cdot \alpha^3 = \alpha^n$.

We show that paths attain the bound from the previous theorem.

Proposition 4 *For positive integers n, let a_n denote the number of minimal double dominating sets of the path P_n. We have*

$$a_n = \begin{cases} 0 & \textit{if } n = 1; \\ 1 & \textit{if } n = 2, 3, 4, 5; \\ a_{n-5} + a_{n-4} + a_{n-3} & \textit{if } n \geq 6. \end{cases}$$

Proof. Obviously, the one-vertex graph has no mdds. It is easy to see that a path on at most five vertices has exactly one mdds. Observe that the path P_6 has two minimal double dominating sets. Now assume that $n \geq 7$. Let $T' = T - v_n - v_{n-1} - v_{n-2}$, $T'' = T' - v_{n-3}$ and $T''' = T'' - v_{n-4}$. It follows from the last paragraph of the proof of Theorem 3 that $a_n = a_{n-5} + a_{n-4} + a_{n-3}$.

Solving the recurrence $a_n = a_{n-5} + a_{n-4} + a_{n-3}$, we get $\lim_{n\to\infty} \sqrt[n]{a_n} = \alpha$, where $\alpha \approx 1.3247$ is the positive solution of the equation $x^3 - x - 1 = 0$ (notice that $x^5 - x^2 - x - 1 = (x^2 + 1)(x^3 - x - 1)$). This implies that the bound from Theorem 3 is tight.

It is an open problem to prove the tightness of an upper bound on the number of minimal dominating sets of a tree. In [10] it has been proved that any tree of order n has less than 1.4656^n minimal dominating sets. A family of trees having more than 1.4167^n minimal dominating sets has also been given.

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