An algorithm for listing all minimal 2-dominating sets of a tree

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Abstract. We provide an algorithm for listing all minimal 2-dominating sets of a tree of order n in time $\mathcal{O}(1.3248^n)$. This implies that every tree has at most 1.3248^n minimal 2-dominating sets. We also show that this bound is tight.

Keywords: domination, 2-domination, minimal 2-dominating set, tree, combinatorial bound, exponential algorithm, listing algorithm

1 Introduction

Let $G = (V, E)$ be a graph. The order of a graph is the number of its vertices. By the neighborhood of a vertex v of G we mean the set $N_G(v) = \{u \in V(G) \colon uv \in E(G)\}$. The degree of a vertex v, denoted by $d_G(v)$, is the cardinality of its neighborhood. By a leaf we mean a vertex of degree one, while a support vertex is a vertex adjacent to a leaf. The distance between two vertices of a graph is the number of edges in a shortest path connecting them. The eccentricity of a vertex is the greatest distance between it and any other vertex. The diameter of a graph G, denoted by $\mathrm{diam}(G)$, is the maximum eccentricity among all vertices of G. Denote by P_n a path on n vertices. By a star we mean a connected graph in which exactly one vertex has degree greater than one.

A subset $D \subseteq V(G)$ is a dominating set of G if every vertex of $V(G) \setminus D$ has a neighbor in D, while it is a 2-dominating set of G if every vertex of $V(G) \setminus D$ has at least two neighbors in D. A dominating (2-dominating, respectively) set D is minimal if no proper subset of D is a dominating (2-dominating, respectively) set of G. A minimal 2-dominating set is abbreviated as m2ds. Note that 2-domination is a type of multiple domination in which each vertex, which is not in the dominating set, is dominated at least k times for a fixed positive integer k. Multiple domination was introduced by Fink and Jacobson [7], and further studied for example in [2, 10, 18]. For a comprehensive survey of domination in graphs, see [11, 12].

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Observation 1 *Every leaf of a graph G is in every 2-dominating set of G.*

One of the typical questions in graph theory is how many subgraphs of a given property can a graph on n vertices have. For example, the famous Moon and Moser theorem [17] says that every graph on n vertices has at most $3^{n/3}$ maximal independent sets.

Combinatorial bounds are of interest not only on their own, but also because they are used for algorithm design as well. Lawler [16] used the Moon-Moser bound on the number of maximal independent sets to construct an $(1+\sqrt[3]{3})^n \cdot n^{\mathcal{O}(1)}$ time graph coloring algorithm, which was the fastest one known for twenty-five years. In 2003 Eppstein [6] reduced the running time of a graph coloring to $\mathcal{O}(2.4151^n)$. In 2006 the running time was reduced [1, 14] to $\mathcal{O}(2^n)$. For an overview of the field, see [9].

Fomin et al. [8] constructed an algorithm for listing all minimal dominating sets of a graph on n vertices in time $\mathcal{O}(1.7159^n)$. There were also given graphs ($n/6$ disjoint copies of the octahedron) having $15^{n/6} \approx 1.5704^n$ minimal dominating sets. This establishes a lower bound on the running time of an algorithm for listing all minimal dominating sets of a given graph.

The number of maximal independent sets in trees was investigated in [19]. Couturier et al. [5] considered minimal dominating sets in various classes of graphs. The authors of [13] investigated the enumeration of minimal dominating sets in graphs.

Bród and Skupień [3] gave bounds on the number of dominating sets of a tree. They also characterized the extremal trees. The authors of [4] investigated the number of minimal dominating sets in trees containing all leaves.

In [15] an algorithm was given for listing all minimal dominating sets of a tree of order n in time $\mathcal{O}(1.4656^n)$, implying that every tree has at most 1.4656^n minimal dominating sets. An infinite family of trees for which the number of minimal dominating sets exceeds 1.4167^n was also given. This established a lower bound on the running time of an algorithm for listing all minimal dominating sets of a given tree.

We provide an algorithm for listing all minimal 2-dominating sets of a tree of order n in time $\mathcal{O}(1.3248^n)$. This implies that every tree has at most 1.3248^n minimal 2-dominating sets. We also show that this bound is tight.

2 Results

We describe an algorithm for listing all minimal 2-dominating sets of a given input tree. We prove that the running time of the algorithm is $\mathcal{O}(1.3248^n)$, implying that every tree has at most 1.3248^n minimal 2-dominating sets.

Theorem 2 *Every tree T of order n has at most α^n minimal 2-dominating sets, where $\alpha \approx 1.32472$ is the positive solution of the equation $x^3 - x - 1 = 0$.*

Proof. In our algorithm, the iterator of the solutions for a tree T is denoted by $\mathcal{F}(T)$. To obtain the upper bound on the number of minimal 2-dominating sets of a tree, we prove that the algorithm lists these sets in time $\mathcal{O}(1.3248^n)$. Notice that the diameter of a tree can easily be determined in polynomial time. If $\mathrm{diam}(T) = 0$, then $T = P_1 = v_1$. Let $\mathcal{F}(T) = \{\{v_1\}\}$. Obviously, $\{v_1\}$ is the only m2ds of the path P_1. We have $n = 1$ and $|\mathcal{F}(T)| = 1$. We also have $1 < \alpha$. If $\mathrm{diam}(T) = 1$, then $T = P_2 = v_1v_2$. Let $\mathcal{F}(T) = \{\{v_1, v_2\}\}$. It is easy to observe that $\{v_1, v_2\}$ is the only m2ds of the path P_2. We have $n = 2$ and $|\mathcal{F}(T)| = 1$. Obviously, $1 < \alpha^2$. If $\mathrm{diam}(T) = 2$, then T is a star. Denote by x the support vertex of T. Let $\mathcal{F}(T) = \{V(T) \setminus \{x\}\}$. It is easy to observe that $V(T) \setminus \{x\}$ is the only m2ds of the tree T. We have $n \geq 3$ and $|\mathcal{F}(T)| = 1$. Obviously, $1 < \alpha^n$.

Now consider trees T with $\mathrm{diam}(T) \geq 3$. The results we obtain by the induction on the number n. Assume that they are true for every tree T' of order $n' < n$. The tree T can easily be rooted at a vertex r of maximum eccentricity $\mathrm{diam}(T)$ in polynomial time. A leaf, say t, at maximum distance from r, can also be easily computed in polynomial time. Let v denote the parent of t and let u denote the parent of v in the rooted tree. If $\mathrm{diam}(T) \geq 4$, then let w denote the parent of u. By T_x we denote the subtree induced by a vertex x and its descendants in the rooted tree T.

If $d_T(v) \geq 3$, then let $T' = T - T_v$ and let T'' differ from T' only in that it has the vertex v. Let $\mathcal{F}(T)$ be as follows,

$$\begin{aligned}&\{D' \cup V(T_v) \setminus \{v\} \colon D' \in \mathcal{F}(T')\}\\ \cup\, &\{D'' \cup V(T_v) \setminus \{v\} \colon D'' \in \mathcal{F}(T'') \text{ and } D'' \setminus \{v\} \notin \mathcal{F}(T')\}.\end{aligned}$$

Let us observe that all elements of $\mathcal{F}(T)$ are minimal 2-dominating sets of the tree T. Now let D be any m2ds of T. Observation 1 implies that $V(T_v) \setminus \{v\} \subseteq D$. If $v \notin D$, then observe that $D \cap V(T')$ is an m2ds of the tree T'. By the inductive hypothesis we have $D \cap V(T') \in \mathcal{F}(T')$. Now assume that $v \in D$. It is easy to observe that $D \cap V(T'')$ is an m2ds of the tree T''. By the inductive hypothesis we have $D \cap V(T'') \in \mathcal{F}(T'')$. The set $D \cap V(T')$ is not an m2ds of the tree T', otherwise $D \setminus \{v\}$ is a 2-dominating set of the tree T, a contradiction to the minimality of D. By the inductive hypothesis we have $D \cap V(T') \notin \mathcal{F}(T')$. Therefore $\mathcal{F}(T)$ contains all minimal 2-dominating sets of the tree T. Now we get $|\mathcal{F}(T)| = |\mathcal{F}(T')| + |\{D'' \in \mathcal{F}(T'') \colon D'' \setminus \{v\} \notin \mathcal{F}(T')\}| \leq |\mathcal{F}(T')| + |\mathcal{F}(T'')| \leq \alpha^{n-3} + \alpha^{n-2} = \alpha^{n-3}(\alpha + 1) = \alpha^{n-3} \cdot \alpha^3 = \alpha^n$.

If $d_T(v) = 2$ and $d_T(u) \geq 3$, then let $T' = T - T_v$, $T'' = T - T_u$, and

$$\mathcal{F}(T) = \{D' \cup \{t\} \colon u \in D' \in \mathcal{F}(T')\} \cup \{D'' \cup V(T_u) \setminus \{u\} \colon D'' \in \mathcal{F}(T'')\}.$$

Let us observe that all elements of $\mathcal{F}(T)$ are minimal 2-dominating sets of the tree T. Now let D be any m2ds of T. By Observation 1 we have $t \in D$. If $v \notin D$, then $u \in D$ as the vertex v has to be dominated twice. Observe that $D \setminus \{t\}$ is an m2ds of the tree T'. By the inductive hypothesis we have $D \setminus \{t\} \in \mathcal{F}(T')$. Now assume that $v \in D$. We have $u \notin D$, otherwise $D \setminus \{v\}$ is a 2-dominating set of the tree T, a contradiction to the minimality of D. Observe that $D \cap V(T'')$ is an m2ds of the tree T''. By the inductive hypothesis we have $D \cap V(T'') \in \mathcal{F}(T'')$. Therefore $\mathcal{F}(T)$ contains all minimal 2-dominating sets of the tree T. Now we get

$|\mathcal{F}(T)| = |\{D' \in \mathcal{F}(T') \colon u \in D'\}| + |\mathcal{F}(T'')| \leq |\mathcal{F}(T')| + |\mathcal{F}(T'')| \leq \alpha^{n-2} + \alpha^{n-3}$ $= \alpha^{n-3}(\alpha+1) = \alpha^{n-3} \cdot \alpha^3 = \alpha^n$.

If $d_T(v) = d_T(u) = 2$, then let $T' = T - T_v$, $T'' = T - T_u$, and

$$\mathcal{F}(T) = \{D' \cup \{t\} \colon D' \in \mathcal{F}(T')\} \cup \{D'' \cup \{v, t\} \colon w \in D'' \in \mathcal{F}(T'')\}.$$

Let us observe that all elements of $\mathcal{F}(T)$ are minimal 2-dominating sets of the tree T. Now let D be any m2ds of T. By Observation 1 we have $t \in D$. If $v \notin D$, then observe that $D \setminus \{t\}$ is an m2ds of the tree T'. By the inductive hypothesis we have $D \setminus \{t\} \in \mathcal{F}(T')$. Now assume that $v \in D$. We have $u \notin D$, otherwise $D \setminus \{v\}$ is a 2-dominating set of the tree T, a contradiction to the minimality of D. Moreover, we have $w \in D$ as the vertex u has to be dominated twice. Observe that $D \setminus \{v, t\}$ is an m2ds of the tree T''. By the inductive hypothesis we have $D \setminus \{v, t\} \in \mathcal{F}(T'')$. Therefore $\mathcal{F}(T)$ contains all minimal 2-dominating sets of the tree T. Now we get $|\mathcal{F}(T)| = |\mathcal{F}(T')| + |\{D'' \in \mathcal{F}(T'') \colon w \in D''\}|$ $\leq |\mathcal{F}(T')| + |\mathcal{F}(T'')| \leq \alpha^{n-2} + \alpha^{n-3} = \alpha^{n-3}(\alpha+1) = \alpha^{n-3} \cdot \alpha^3 = \alpha^n$.

We now show that paths attain the bound from the previous theorem.

Proposition 3 *For positive integers n, let a_n denote the number of minimal 2-dominating sets of the path P_n. We have*

$$a_n = \begin{cases} 1 & \text{if } n \leq 3; \\ a_{n-3} + a_{n-2} & \text{if } n \geq 4. \end{cases}$$

Proof. It is easy to see that a path on at most three vertices has exactly one minimal 2-dominating set. Now assume that $n \geq 4$. Let $T' = T - v_n - v_{n-1}$ and $T'' = T' - v_{n-2}$. It follows from the last paragraph of the proof of Theorem 2 that $a_n = a_{n-3} + a_{n-2}$.

Solving the recurrence $a_n = a_{n-3} + a_{n-2}$, we get $\lim_{n\to\infty} \sqrt[n]{a_n} = \alpha$, where $\alpha \approx 1.3247$ is the positive solution of the equation $x^3 - x - 1 = 0$. This implies that the bound from Theorem 2 is tight.

It is an open problem to prove the tightness of an upper bound on the number of minimal dominating sets of a tree. In [15] it has been proved that any tree of order n has less than 1.4656^n minimal dominating sets. A family of trees having more than 1.4167^n minimal dominating sets has also been given.

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