An Algorithm for Listing all Minimal Double Dominating Sets of a Tree

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Abstract. We provide an algorithm for listing all minimal double dominating sets of a tree of order $n$ in time $O(1.3248^n)$. This implies that every tree has at most $1.3248^n$ minimal double dominating sets. We also show that this bound is tight.

Keywords: domination, double domination, minimal double dominating set, tree, combinatorial bound, exact exponential algorithm, listing algorithm

1. Introduction

Let $G = (V, E)$ be a graph. The order of a graph is the number of its vertices. By the neighborhood of a vertex $v$ of $G$ we mean the set $N_G(v) = \{u \in V(G): uv \in E(G)\}$. The degree of a vertex $v$, denoted by $d_G(v)$, is the cardinality of its neighborhood. By a leaf we mean a vertex of degree one, while a support vertex is a vertex adjacent to a leaf. We say that a support vertex is strong (weak, respectively) if it is adjacent to at least two leaves (exactly one leaf, respectively). The distance between two vertices of a graph is the number of edges in a shortest path connecting them. The eccentricity of a vertex is the greatest distance between it and any other vertex. The diameter of a graph $G$, denoted by $\text{diam}(G)$, is the maximum eccentricity among all vertices of $G$. A path on $n$ vertices we denote by $P_n$.

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A vertex of a graph is said to dominate itself and all of its neighbors. A subset \( D \subseteq V(G) \) is a dominating set of \( G \) if every vertex of \( G \) is dominated by at least one vertex of \( D \), while it is a double dominating set of \( G \) if every vertex of \( G \) is dominated by at least two vertices of \( D \). A dominating (double dominating, respectively) set \( D \) is minimal if no proper subset of \( D \) is a dominating (double dominating, respectively) set of \( G \). A minimal double dominating set is abbreviated as mdds.

Double domination in graphs was introduced by Harary and Haynes [6]. For a comprehensive survey of domination in graphs, see [7, 8].

**Observation 1.** Every leaf of a graph \( G \) is in every double dominating set of \( G \).

**Observation 2.** Every support vertex of a graph \( G \) is in every double dominating set of \( G \).

One of the typical questions in graph theory is how many subgraphs of a given property can a graph on \( n \) vertices have. For example, the famous Moon and Moser theorem [12] says that every graph on \( n \) vertices has at most \( 3^{n/3} \) maximal independent sets.

Combinatorial bounds are of interest not only on their own, but also because they are used for algorithm design as well. Lawler [11] used the Moon-Moser bound on the number of maximal independent sets to construct an \((1 + \sqrt{3})^n \cdot n^{O(1)}\) time graph coloring algorithm, which was the fastest one known for twenty-five years. For an overview of the field, see [5].

Fomin et al. [4] constructed an algorithm for listing all minimal dominating sets of a graph on \( n \) vertices in time \( O(1.7159^n) \). There were also given graphs (\( n/6 \) disjoint copies of the octahedron) having \( 15^{n/6} \approx 1.5704^n \) minimal dominating sets. This establishes a lower bound on the running time of an algorithm for listing all minimal dominating sets of a given graph.

The number of maximal independent sets in trees was investigated in [13]. Couturier et al. [3] considered minimal dominating sets in various classes of graphs. The authors of [9] investigated the enumeration of minimal dominating sets in graphs.

Bród and Skupień [1] gave bounds on the number of dominating sets of a tree. They also characterized the extremal trees. The authors of [2] investigated the number of minimal dominating sets in trees containing all leaves.

In [10] an algorithm was given for listing all minimal dominating sets of a tree of order \( n \) in time \( O(1.4656^n) \). This implies that every tree has at most \( 1.4656^n \) minimal dominating sets. An infinite family of trees for which the number of minimal dominating sets exceeds \( 1.4167^n \) was also given. This establishes a lower bound on the running time of an algorithm for listing all minimal dominating sets of a given tree.

We provide an algorithm for listing all minimal double dominating sets of a tree of order \( n \) in time \( O(1.3248^n) \). This implies that every tree has at most \( 1.3248^n \) minimal double dominating sets. We also show that this bound is tight.

### 2. Results

We describe a recursive algorithm which lists all minimal double dominating sets of a given input tree. We prove that the running time of this algorithm is \( O(1.3248^n) \), implying that every tree has at most \( 1.3248^n \) minimal double dominating sets.
Theorem 3. Every tree $T$ of order $n$ has at most $\alpha^n$ minimal double dominating sets, where $\alpha \approx 1.32472$ is the positive solution of the equation $x^3 - x - 1 = 0$, and all those sets can be listed in time $O(1.3248^n)$.

Proof:
The family of sets returned by our algorithm is denoted by $F(T)$. To obtain the upper bound on the number of minimal double dominating sets of a tree, we prove that the algorithm lists these sets in time $O(1.3248^n)$. If $\text{diam}(T) \leq 3$, then let $F(T) = \{ V(T) \}$. Every vertex of $T$ is a leaf or a support vertex. Observations 1 and 2 imply that $V(T)$ is the only mdds of $T$. We have $n \geq 2$ and $|F(T)| = 1$. Obviously, $1 < \alpha^n$.

Now assume that $\text{diam}(T) \geq 4$. Thus the order $n$ of the tree $T$ is at least five. The results we obtain by the induction on the number $n$. Assume that they are true for every tree $T'$ of order $n' < n$.

First assume that some support vertex of $T$, say $x$, is strong. Let $y$ and $z$ be leaves adjacent to $x$. Let $T' = T - y$, and let

$$F(T) = \{ D' \cup \{ y \} : D' \in F(T') \}.$$ 

Let $D'$ be an mdds of the tree $T'$. By Observation 2 we have $x \in D'$. It is easy to see that $D' \cup \{ y \}$ is an mdds of $T$. Thus all elements of the family $F(T)$ are minimal double dominating sets of the tree $T$. Now let $D$ be any mdds of the tree $T$. By Observations 1 and 2 we have $x, y, z \in D$. Let us observe that $D \setminus \{ y \}$ is an mdds of the tree $T'$ as the vertex $x$ is still dominated at least twice. By the inductive hypothesis we have $D \setminus \{ y \} \in F(T')$. Therefore the family $F(T)$ contains all minimal double dominating sets of the tree $T$. Now we get $|F(T)| = |F(T')| \leq \alpha^{n-1} < \alpha^n$. Henceforth, we can assume that every support vertex of $T$ is weak.

We now root $T$ at a vertex $r$ of maximum eccentricity $\text{diam}(T)$. Let $t$ be a leaf at maximum distance from $r, v$ be the parent of $t, u$ be the parent of $v$, and $w$ be the parent of $u$ in the rooted tree. If $\text{diam}(T) \geq 5$, then let $d$ be the parent of $w$. By $T_x$ we denote the subtree induced by a vertex $x$ and its descendants in the rooted tree $T$.

Assume that $u$ is adjacent to a leaf, say $x$. Let $T' = T - T_u$, and let

$$F(T) = \{ D' \cup \{ v, t \} : D' \in F(T') \}.$$ 

Let us observe that all elements of the family $F(T)$ are minimal double dominating sets of the tree $T$. Now let $D$ be any mdds of the tree $T$. By Observations 1 and 2 we have $t, x, v, u \in D$. It is easy to observe that $D \setminus \{ v, t \}$ is an mdds of the tree $T'$. By the inductive hypothesis we have $D \setminus \{ v, t \} \in F(T')$. Therefore the family $F(T)$ contains all minimal double dominating sets of the tree $T$. Now we get $|F(T)| = |F(T')| \leq \alpha^{n-2} < \alpha^n$.

Now assume that all children of $u$ are support vertices. Assume that $d_T(u) \geq 4$. Let $T' = T - T_v$, and let

$$F(T) = \{ D' \cup \{ v, t \} : D' \in F(T') \}.$$ 

Let us observe that all elements of the family $F(T)$ are minimal double dominating sets of the tree $T$. Now let $D$ be any mdds of the tree $T$. By Observations 1 and 2 we have $v, t \in D$. Let us observe that $D \setminus \{ v, t \}$ is an mdds of the tree $T'$ as the vertex $u$ is still dominated at least twice. By the inductive hypothesis we have $D \setminus \{ v, t \} \in F(T')$. Therefore the family $F(T)$ contains all minimal double dominating sets of the tree $T$. Now we get $|F(T)| = |F(T')| \leq \alpha^{n-2} < \alpha^n$. 

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Now assume that \( d_T(u) = 3 \). Let \( x \) be the child of \( u \) other than \( v \). The leaf adjacent to \( x \) we denote by \( y \). Let \( T' = T - T_u \) and \( T'' = T - T_y - y \). Let \( \mathcal{F}(T) \) be a family as follows,

\[
\{ D' \cup \{ t, v, x, y \} : D' \in \mathcal{F}(T') \}
\cup \{ D'' \cup \{ v, t, y \} : D'' \in \mathcal{F}(T'') \text{ and } D'' \setminus \{ u, x \} \notin \mathcal{F}(T') \}.
\]

Let us observe that all elements of the family \( \mathcal{F}(T) \) are minimal double dominating sets of the tree \( T \). Now let \( D \) be any mdds of the tree \( T \). By Observations 1 and 2 we have \( v, t, x, y \in D \). If \( u \notin D \), then observe that \( D \setminus \{ v, t, x, y \} \) is an mdds of the tree \( T' \). By the inductive hypothesis we have \( D \setminus \{ v, t, x, y \} \notin \mathcal{F}(T') \). Now assume that \( u \in D \). It is easy to observe that \( D \setminus \{ v, t, y \} \) is an mdds of the tree \( T'' \). By the inductive hypothesis we have \( D \setminus \{ v, t, y \} \notin \mathcal{F}(T'') \). Let us observe that \( D \setminus \{ u, v, t, y \} \) is not a double dominating set of the tree \( T' \), otherwise \( D \setminus \{ u \} \) is a double dominating set of the tree \( T \), a contradiction to the minimality of \( D \). Therefore the family \( \mathcal{F}(T) \) contains all minimal double dominating sets of the tree \( T \). Now we get \( |\mathcal{F}(T)| = |\mathcal{F}(T')| + |\mathcal{F}(T'')| \) such that \( |\mathcal{F}(T'')| \leq \alpha^{n-5} + \alpha^{n-3} = \alpha^{n-5}(\alpha^2 + 1) < \alpha^{n-5} \cdot \alpha^5 = \alpha^n \).

Now assume that \( d_T(u) = 2 \). Assume that \( d_T(w) \geq 3 \). First assume that \( w \) is adjacent to a leaf, say \( k \). Let \( T' = T - T_u \), and let

\[
\mathcal{F}(T) = \{ D' \cup \{ v, t \} : D' \in \mathcal{F}(T') \}.
\]

Let us observe that all elements of the family \( \mathcal{F}(T) \) are minimal double dominating sets of the tree \( T \). Now let \( D \) be any mdds of the tree \( T \). By Observations 1 and 2 we have \( v, t, k, l \in D \). If \( u \notin D \), then \( D \setminus \{ v, t \} \) is a minimal double dominating set of the tree \( T' \). By the inductive hypothesis we have \( D \setminus \{ v, t \} \notin \mathcal{F}(T') \). Therefore the family \( \mathcal{F}(T) \) contains all minimal double dominating sets of the tree \( T \). Now we get \( |\mathcal{F}(T)| = |\mathcal{F}(T')| \leq \alpha^{n-3} < \alpha^n \).

Now assume that there is a child of \( w \), say \( k \), such that the distance of \( w \) to the most distant vertex of \( T_k \) is two. Thus \( k \) is a support vertex of degree two. The leaf adjacent to \( k \) we denote by \( l \). Let \( T' = T - T_u - l \) and \( T'' = T - T_w \). Let

\[
\mathcal{F}(T) = \{ D' \cup \{ v, t, l \} : D' \in \mathcal{F}(T') \} \cup \{ D'' \cup V(T_w) \setminus \{ w \} : D'' \in \mathcal{F}(T'') \}.
\]

Let us observe that all elements of the family \( \mathcal{F}(T) \) are minimal double dominating sets of the tree \( T \). Now let \( D \) be any mdds of the tree \( T \). By Observations 1 and 2 we have \( v, t, k, l \in D \). If \( u \notin D \), then \( w \in D \) as the vertex \( u \) has to be dominated twice. It is easy to observe that \( D \setminus \{ v, t, l \} \) is an mdds of the tree \( T'' \). By the inductive hypothesis we have \( D \setminus \{ v, t, l \} \notin \mathcal{F}(T'') \). Now assume that \( u \in D \). We have \( w \notin D \), otherwise \( D \setminus \{ u \} \) is a double dominating set of the tree \( T \), a contradiction to the minimality of \( D \). Observe that \( D \cap V(T'' \setminus \{ w \}) \) is an mdds of the tree \( T'' \). By the inductive hypothesis we have \( D \cap V(T'' \setminus \{ w \}) \notin \mathcal{F}(T'') \). Therefore the family \( \mathcal{F}(T) \) contains all minimal double dominating sets of the tree \( T \). Now we get \( |\mathcal{F}(T)| = |\mathcal{F}(T')| + |\mathcal{F}(T'')| \leq \alpha^{n-4} + \alpha^{n-6} = \alpha^{n-5}(\alpha + 1) < \alpha^{n-5} \cdot \alpha^3 = \alpha^n \).

Now assume that for every child of \( w \), say \( k \), the distance of \( w \) to the most distant vertex of \( T_k \) is three. Due to the earlier analysis of the degree of the vertex \( u \), which is a child of \( w \), it suffices to consider only the possibility when \( T_k \) is a path \( P_3 \). Let \( T' = T - T_w \). Let \( T''(T''') \), respectively be a tree that differs from \( T' \) only in that it has the vertex \( w \) (the vertices \( w \) and \( u \), respectively). Let \( \mathcal{F}(T) \) be a family
as follows,

\[ \{D' \cup V(T_w) \setminus \{w\}: D' \in F(T')\} \]

\[ \cup \{D'' \cup V(T_w) \setminus \{N_T(w) \setminus \{d\}\}: D'' \in F(T'')\} \]

\[ \cup \{D''' \cup V(T_w) \setminus \{N_T(w) \setminus \{x\}\}: \quad \text{if } d \notin D''' \in F(T'''\prime) \text{ and } x \in N_T(w) \setminus \{d\}\}. \]

Let us observe that all elements of the family \( F(T) \) are minimal double dominating sets of the tree \( T \). Now let \( D \) be any mdds of the tree \( T \). If \( w \notin D \), then observe that \( D \cap V(T') \) is a mdds of the tree \( T' \). By the inductive hypothesis we have \( D \cap V(T') \in F(T') \). Now assume that \( w \in D \). If no child of \( w \) belongs to the set \( D \), then observe that \( D \cap V(T''\prime) \) is an mdds of the tree \( T''\prime \). By the inductive hypothesis we have \( D \cap V(T''\prime) \in F(T''\prime) \). Now assume that some child of \( w \), say \( x \), belongs to the set \( D \). Let us observe that \( (D \cup \{w\}) \cap V(T''\prime) \) is an mdds of the tree \( T''\prime \). By the inductive hypothesis we have \( (D \cup \{w\}) \cap V(T''\prime) \in F(T''\prime) \). Therefore the family \( F(T) \) contains all minimal double dominating sets of the tree \( T \). Now we get \( |F(T)| = |F(T')| + |F(T'')| + (d_T(w) - 1) \cdot \{D''' \cap F(T''\prime): d \notin D'''\} \)

\[ \leq |F(T')| + |F(T'')| + (d_T(w) - 1) \cdot \{D''' \cap F(T'')\} \]

\[ \leq \alpha_n^{n - 3d_T(w) + 2} + \alpha_n^{n - 3d_T(w) + 3} + (d_T(w) - 1) \cdot \alpha_n^{n - 3d_T(w) + 4} \]

To show that \( \alpha_n^{n - 3d_T(w) + 2} + \alpha_n^{n - 3d_T(w) + 3} + (d_T(w) - 1) \cdot \alpha_n^{n - 3d_T(w) + 4} < \alpha^6 \), it suffices to show that \( \alpha^2 + \alpha^3 + (d_T(w) - 1) \cdot \alpha^4 < \alpha^{3d_T(w)} \). We prove this by the induction on the degree of the vertex \( w \). For \( d_T(w) = 3 \) we have \( \alpha^2 + \alpha^3 + (d_T(w) - 1) \cdot \alpha^4 = 2\alpha^4 + \alpha^3 + \alpha^2 = 2\alpha^4 + \alpha^2(\alpha + 1) = 2\alpha^4 + \alpha^5 = \alpha^4(\alpha + 1) + \alpha^4 = \alpha^7 + \alpha^4 = \alpha^6(\alpha^3 - 1) + \alpha^4 = \alpha^9 + \alpha^4 - \alpha^6 < \alpha^9 = \alpha^3d_T(w) \). Now we prove that the inequality \( \alpha^2 + \alpha^3 + (k - 1) \cdot \alpha^4 < \alpha^{3k} \) is satisfied for an integer \( k = d_T(w) \geq 3 \), then it is also satisfied for \( k + 1 \). We have \( \alpha^2 + \alpha^3 + k\alpha^4 = \alpha^2 + \alpha^3 + (k - 1) \cdot \alpha^4 + \alpha^4 < \alpha^{3k} + \alpha^{3k+1} = \alpha^{3k+3} \).

Now assume that \( d_T(w) = 2 \). If \( d_T(d) = 1 \), then let \( F(T) = \{\{d, w, v, t\}\} \). The tree \( T \) is a path \( P_6 \).

It is easy to observe that \( \{d, w, v, t\} \) is the only mdds of the tree \( T \). We have \( n = 5 \) and \( |F(T)| = 1 \). Obviously, \( 1 < \alpha^5 \). Now assume that \( d_T(d) \geq 2 \). Due to the earlier analysis of the degrees of the vertices \( w \) and \( u \), we may assume that for every child of \( d \), say \( k \), the tree \( T_k \) is a path on at most four vertices. Let \( T' = T - T_u \), \( T'' = T - T_w \) and \( T''' = T - T_d \). If \( T''' \) is a single vertex, then let \( F(T) = \{\{r, d, w, v, t\}, \{r, d, u, v, t\}\} \). The tree \( T \) is a path \( P_6 \). Let us observe that \( \{r, d, w, v, t\} \) and \( \{r, d, u, v, t\} \) are the only two minimal double dominating sets of the tree \( T \). We have \( n = 6 \) and \( |F(T)| = 2 \). Obviously, \( 2 < \alpha^6 \). Now assume that \( |V(T''\prime)\| \geq 2 \). Let \( F(T) \) be a family as follows,

\[ \{D' \cup \{v, t\}: D' \in F(T')\} \]

\[ \cup \{D'' \cup \{u, v, t\}: d \in D'' \in F(T'')\} \]

\[ \cup \{D''' \cup V(T_d) \setminus \{d\}: D''' \in F(T''\prime)\}, \]

where the third component is ignored if \( d \) is adjacent to a leaf. Let us observe that all elements of the family \( F(T) \) are minimal double dominating sets of the tree \( T \). Now let \( D \) be any mdds of the tree \( T \). By Observations 1 and 2 we have \( v, t \in D \). If \( u \notin D \), then observe that \( D \setminus \{v, t\} \) is an mdds of the tree \( T' \). By the inductive hypothesis we have \( D \setminus \{v, t\} \in F(T') \). Now assume that \( u \in D \). If \( w \notin D \), then observe that \( D \setminus \{u, v, t\} \) is an mdds of the tree \( T'' \). By the inductive hypothesis we have \( D \setminus \{u, v, t\} \in F(T'') \). Now assume that \( w \in D \). We have \( d \notin D \), otherwise \( D \setminus \{u\} \) is a double dominating set of the tree \( T \), a contradiction to the minimality of \( D \). Observe that \( D \cap V(T''\prime) \) is an mdds of the tree \( T'' \). By the inductive hypothesis we have \( D \cap V(T''\prime) \in F(T''\prime) \). Therefore the family \( F(T) \) contains all minimal double dominating sets of the tree \( T \). Now we get \( |F(T)| = |F(T')| + |F(T'')| + |F(T'''\prime)| \)

\[ \leq n - \frac{3}{2} + \frac{n}{2} + \alpha_n^{n - 3} + \alpha_n^{n - 4} + \alpha_n^{n - 5} \]

\[ = \alpha_n^{n - 5}(\alpha^2 + \alpha + 1) = \alpha_n^{n - 5}(\alpha^2 + \alpha^3) = \alpha_n^{n - 3}(\alpha + 1) = \alpha_n^{n - 3} \cdot \alpha^3 = \alpha^n. \]
We show that paths attain the bound from the previous theorem.

**Proposition 4.** For positive integers \( n \), let \( a_n \) denote the number of minimal double dominating sets of the path \( P_n \). We have

\[
a_n = \begin{cases} 
0 & \text{if } n = 1; \\
1 & \text{if } n = 2, 3, 4, 5; \\
a_{n-5} + a_{n-4} + a_{n-3} & \text{if } n \geq 6.
\end{cases}
\]

**Proof:**

Obviously, the one-vertex graph has no mdds. It is easy to see that a path on at most five vertices has exactly one mdds. Observe that the path \( P_6 \) has two minimal double dominating sets. Now assume that \( n \geq 7 \). Let \( T' = T - v_{n} - v_{n-1} - v_{n-2}, T'' = T' - v_{n-3} \) and \( T'' = T'' - v_{n-4} \). It follows from the last paragraph of the proof of Theorem 3 that \( a_n = a_{n-5} + a_{n-4} + a_{n-3} \).

Solving the recurrence \( a_n = a_{n-5} + a_{n-4} + a_{n-3} \), we get \( \lim_{n \to \infty} \sqrt[n]{a_n} = \alpha \), where \( \alpha \approx 1.3247 \) is the positive solution of the equation \( x^3 - x - 1 = 0 \) (notice that \( x^5 - x^2 - x - 1 = (x^2 + 1)(x^3 - x - 1) \)). This implies that the bound from Theorem 3 is tight.

It is an open problem to prove the tightness of an upper bound on the number of minimal dominating sets of a tree. In [10] it has been proved that any tree of order \( n \) has less than \( 1.4656^n \) minimal dominating sets. A family of trees having more than \( 1.4167^n \) minimal dominating sets has also been given.

**References**


