An upper bound for the double outer-independent domination number of a tree

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Abstract

A vertex of a graph is said to dominate itself and all of its neighbors. A double outer-independent dominating set of a graph \( G \) is a set \( D \) of vertices of \( G \) such that every vertex of \( G \) is dominated by at least two vertices of \( D \), and the set \( V(G) \setminus D \) is independent. The double outer-independent domination number of a graph \( G \), denoted by \( \gamma_{oi}^d(G) \), is the minimum cardinality of a double outer-independent dominating set of \( G \). We prove that for every nontrivial tree \( T \) of order \( n \), with \( l \) leaves and \( s \) support vertices we have \( \gamma_{oi}^d(T) \leq (2n + l + s)/3 \), and we characterize the trees attaining this upper bound.

Keywords: double outer-independent domination, double domination, tree.

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1 Introduction

Let \( G = (V, E) \) be a graph. By the neighborhood of a vertex \( v \) of \( G \) we mean the set \( N_G(v) = \{ u \in V(G) : uv \in E(G) \} \). The degree of a vertex \( v \), denoted by \( d_G(v) \), is the cardinality of its neighborhood. By a leaf we mean a vertex of degree one, while a support vertex is a vertex adjacent to a leaf. We say that a support vertex is strong (weak, respectively) if it is adjacent to at least two leaves (exactly one leaf, respectively). The path on \( n \) vertices we denote by \( P_n \). We say that a subset of \( V(G) \) is independent if there is no edge between any two vertices of this set.

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A vertex of a graph is said to dominate itself and all of its neighbors. A subset \( D \subseteq V(G) \) is a dominating set of \( G \) if every vertex of \( G \) is dominated by at least one vertex of \( D \), while it is a double dominating set of \( G \) if every vertex of \( G \) is dominated by at least two vertices of \( D \). The domination (double domination, respectively) number of \( G \), denoted by \( \gamma(G) \) (\( \gamma_d(G) \), respectively), is the minimum cardinality of a dominating (double dominating, respectively) set of \( G \). Double domination in graphs was introduced by Harary and Haynes [4], and further studied for example in [1, 3]. For a comprehensive survey of domination in graphs, see [5, 6].

A subset \( D \subseteq V(G) \) is a double outer-independent dominating set, abbreviated DOIDS, of \( G \) if every vertex of \( G \) is dominated by at least two vertices of \( D \), and the set \( V(G) \setminus D \) is independent. The double outer-independent domination number of a graph \( G \), denoted by \( \gamma_{oi}(G) \), is the minimum cardinality of a double outer-independent dominating set of \( G \). A double outer-independent dominating set of \( G \) of minimum cardinality is called a \( \gamma_{oi}(G) \)-set. The study of double outer-independent domination in graphs was initiated in [7].

A 2-dominating set of a graph \( G \) is a set \( D \) of vertices of \( G \) such that every vertex of \( V(G) \setminus D \) has at least two neighbors in \( D \). The 2-domination number of \( G \), denoted by \( \gamma_2(G) \), is the minimum cardinality of a 2-dominating set of \( G \). Blidia, Chellali, and Favaron [2] proved the following upper bound on the 2-domination number of a tree. For every nontrivial tree \( T \) of order \( n \) with \( l \) leaves we have \( \gamma_2(T) \leq (n + l)/2 \). They also characterized the extremal trees.

We prove the following upper bound on the double outer-independent domination number of a tree. For every nontrivial tree \( T \) of order \( n \), with \( l \) leaves and \( s \) support vertices we have \( \gamma_{oi}(T) \leq (2n + l + s)/3 \). We also characterize the trees attaining this upper bound.

## 2 Results

Since the one-vertex graph does not have a double outer-independent dominating set, in this paper, by a tree we mean only a connected graph with no cycle, and which has at least two vertices.

We begin with the following two straightforward observations.

**Observation 1** Every leaf of a graph \( G \) is in every \( \gamma_d(G) \)-set.

**Observation 2** Every support vertex of a graph \( G \) is in every \( \gamma_d(G) \)-set.

We show that if \( T \) is a nontrivial tree of order \( n \), with \( l \) leaves and \( s \) support vertices, then \( \gamma_{oi}(T) \) is bounded above by \( (2n + l + s)/3 \). For the purpose of characterizing the trees attaining this bound we introduce a family \( \mathcal{T} \) of trees \( T = T_k \) that can be obtained as follows. Let \( T_1 \) be a path \( P_3 \) with leaves labeled \( x \) and \( z \), and the support vertex labeled \( y \). Let \( A(T_1) = \{x, y, z\} \). Let \( H_1 \) be a path \( P_2 \)
with vertices labeled $u$ and $v$. Let finally $H_2$ be a path $P_3$ with leaves labeled $u$ and $w$, and the support vertex labeled $v$. If $k$ is a positive integer, then $T_{k+1}$ can be obtained recursively from $T_k$ by one of the following operations.

- Operation $O_1$: Attach a vertex, say $z$, by joining it to a support vertex of $T_k$. Let $A(T_{k+1}) = A(T_k) \cup \{z\}$.

- Operation $O_2$: Attach a vertex, say $z$, by joining it to a leaf of $T_k$ adjacent to a strong support vertex. Let $A(T_{k+1}) = A(T_k) \cup \{z\}$.

- Operation $O_3$: Attach a copy of $H_1$ by joining the vertex $u$ to a vertex of $T_k$ which is not a leaf and is adjacent to a support vertex. Let $A(T_{k+1}) = A(T_k) \cup \{u,v\}$.

- Operation $O_4$: Attach a copy of $H_2$ by joining the vertex $u$ to a leaf of $T_k$ adjacent to a weak support vertex. Let $A(T_{k+1}) = A(T_k) \cup \{v,w\}$.

We now prove that for every tree $T$ of the family $\mathcal{T}$, the set $A(T)$ defined above is a DOIDS of minimum cardinality equal to $(2n + l + s)/3$.

**Lemma 3** If $T \in \mathcal{T}$, then the set $A(T)$ defined above is a $\gamma_d^a(T)$-set of size $(2n + l + s)/3$.

**Proof.** We use the terminology of the construction of the trees $T = T_k$, the set $A(T)$, and the graphs $H_1$ and $H_2$ defined above. To show that $A(T)$ is a $\gamma_d^a(T)$-set of cardinality $(2n + l + s)/3$ we use the induction on the number $k$ of operations performed to construct the tree $T$. If $T = T_1 = P_3$, then $(2n + l + s)/3 = (6 + 2 + 1)/3 = 3 = |A(T)| = \gamma_d^a(T)$. Let $k \geq 2$ be an integer. Assume that the result is true for every tree $T' = T_k$ of the family $\mathcal{T}$ constructed by $k - 1$ operations. For a given tree $T'$, let $n'$ denote its order, $l'$ the number of its leaves, and $s'$ the number of support vertices. Let $T = T_{k+1}$ be a tree of the family $\mathcal{T}$ constructed by $k$ operations.

First assume that $T$ is obtained from $T'$ by operation $O_1$. We have $n = n' + 1$, $l = l' + 1$ and $s = s'$. The vertex to which is attached $z$ we denote by $x$. Let $y$ be a leaf adjacent to $x$ and different from $z$. By Observation 2 we have $x \in A(T')$. It is easy to see that $A(T) = A(T') \cup \{z\}$ is a DOIDS of the tree $T$. Thus $\gamma_d^a(T) \leq \gamma_d^a(T') + 1$. Now let $D$ be any $\gamma_d^a(T)$-set. By Observations 1 and 2 we have $z, y, x \in D$. It is easy to see that $D \setminus \{z\}$ is a DOIDS of the tree $T'$. Therefore $\gamma_d^a(T') \leq \gamma_d^a(T) - 1$. We now conclude that $\gamma_d^a(T) = \gamma_d^a(T') + 1$. We get $\gamma_d^a(T) = |A(T)| = |A(T')| + 1 = (2n' + l' + s')/3 + 1 = (2n - 2 + l - 1 + s)/3 + 1 = (2n + l + s)/3$.

Now suppose that $T$ is obtained from $T'$ by operation $O_2$. We have $n = n' + 1$, $l = l'$ and $s = s' + 1$. The leaf to which is attached $z$ we denote by $x$. By $y$ we denote the neighbor of $x$ other than $z$. By Observation 1 we have $x \in A(T')$. 


It is easy to see that $A(T) = A(T') \cup \{z\}$ is a DOIDS of the tree $T$. Thus $\gamma_d^a(T) \leq \gamma_d^a(T') + 1$. Now let $D$ be any $\gamma_d^a(T)$-set. By Observations 1 and 2 we have $z, x, y \in D$. It is easy to see that $D \setminus \{z\}$ is a DOIDS of the tree $T'$. Therefore $\gamma_d^a(T') \leq \gamma_d^a(T) - 1$. We now conclude that $\gamma_d^a(T) = \gamma_d^a(T') + 1$. We get $\gamma_d^a(T) = |A(T)| = |A(T')| + 1 = (2n' + l' + s')/3 + 1 = (2n - 2 + l + s - 1)/3 + 1 = (2n + l + s)/3$.

Now assume that $T$ is obtained from $T'$ by operation $O_3$. We have $n = n' + 2$, $l = l' + 1$ and $s = s' + 1$. The vertex to which is attached $P_2$ we denote by $x$. Let $y$ be a support vertex adjacent to $x$, and let $z$ be a leaf adjacent to $y$. Obviously, $A(T) = A(T') \cup \{u, v\}$ is a DOIDS of the tree $T$. Thus $\gamma_d^a(T) \leq \gamma_d^a(T') + 2$. Now let $D$ be any $\gamma_d^a(T)$-set. By Observations 1 and 2 we have $v, z, u, y \in D$. If $x \in D$, then it is easy to see that $D \setminus \{u, v\}$ is a DOIDS of the tree $T'$. Now suppose that $x \notin D$. Let $a$ denote a neighbor of $x$ other than $u$ and $y$. The set $V(T) \setminus D$ is independent, thus $a \in D$. Let us observe that now also $D \setminus \{u, v\}$ is a DOIDS of the tree $T'$ as the vertex $x$ is still dominated at least twice. Therefore $\gamma_d^a(T') \leq \gamma_d^a(T) - 2$. We now conclude that $\gamma_d^a(T) = \gamma_d^a(T') + 2$. We get $\gamma_d^a(T) = |A(T)| = |A(T')| + 2 = (2n' + l' + s')/3 + 2 = (2n - 4 + l - 1 + s - 1)/3 + 2 = (2n + l + s)/3$.

Now assume that $T$ is obtained from $T'$ by operation $O_4$. We have $n = n' + 3$, $l = l' + 1$ and $s = s'$. The leaf to which is attached $P_3$ we denote by $x$. By Observation 1 we have $x \in A(T')$. It is easy to see that $D' \cup \{v, w\}$ is a DOIDS of the tree $T$. Thus $\gamma_d^a(T) \leq \gamma_d^a(T') + 2$. Now let us observe that there exists a $\gamma_d^a(T)$-set that does not contain the vertex $u$. Let $D$ be such a set. By Observations 1 and 2 we have $w, v \in D$. Observe that $D \setminus \{v, w\}$ is a DOIDS of the tree $T'$. Therefore $\gamma_d^a(T') \leq \gamma_d^a(T) - 2$. We now conclude that $\gamma_d^a(T) = \gamma_d^a(T') + 2$. We get $\gamma_d^a(T) = |A(T)| = |A(T')| + 2 = (2n' + l' + s')/3 + 2 = (2n - 6 + l + s)/3 + 2 = (2n + l + s)/3$.

We now establish the main result, an upper bound on the double outer-independent domination number of a tree together with the characterization of the extremal trees.

**Theorem 4** If $T$ is a tree of order $n$, with $l$ leaves and $s$ support vertices, then $\gamma_d^a(T) \leq (2n + l + s)/3$ with equality if and only if $T \in \mathcal{T}$.

**Proof.** If $\text{diam}(T) = 1$, then $T = P_2$. We have $\gamma_d^a(T) = 2 < (4 + 2 + 2)/3 = (2n + l + s)/3$. Now suppose that $\text{diam}(T) \geq 2$. Thus the order $n$ of the tree $T$ is at least three. The result we obtain by the induction on the number $n$. Assume that the theorem is true for every tree $T'$ of order $n' < n$, with $l'$ leaves and $s'$ support vertices.

First suppose that some support vertex of $T$, say $x$, is strong. Let $y$ and $z$ be leaves adjacent to $x$. Let $T' = T - y$. We have $n' = n - 1$, $l' = l - 1$ and $s' = s$. Let $D'$ be any $\gamma_d^a(T')$-set. By Observation 2 we have $x \in D'$. It is easy to see
that $D' \cup \{y\}$ is a DOIDS of the tree $T$. Thus $\gamma_d^a(T) \leq \gamma_d^a(T') + 1$. We now get $\gamma_d^a(T) < \gamma_d^a(T') + 1 \leq (2n' + l' + s')/3 + 1 = (2n - 2 + l - 1 + s)/3 + 1 = (2n + l + s)/3$. If $\gamma_d^a(T) = (2n + l + s)/3$, then obviously $\gamma_d^a(T') = (2n' + l' + s')/3$. By the inductive hypothesis we have $T' \in \mathcal{T}$. The tree $T$ can be obtained from $T'$ by operation $O_1$. Thus $T \in \mathcal{T}$. Henceforth, we can assume that every support vertex of $T$ is weak.

We now root $T$ at a vertex $r$ of maximum eccentricity $\text{diam}(T)$. Let $t$ be a leaf at maximum distance from $r$, and let $v$ be the parent of $t$ in the rooted tree. If $\text{diam}(T) \geq 3$, then let $u$ be the parent of $v$. If $\text{diam}(T) \geq 4$, then let $w$ be the parent of $u$. If $\text{diam}(T) \geq 5$, then let $d$ be the parent of $w$. By $T_x$ let us denote the subtree induced by a vertex $x$ and its descendants in the rooted tree $T$.

First suppose that $d_T(u) \geq 3$. Assume that among the children of $u$ there is a support vertex, say $x$, different from $v$. The leaf adjacent to $x$ we denote by $y$. Let $T' = T - T_u$. We have $n' = n - 2$, $l' = l - 1$ and $s' = s - 1$. Let $D'$ be any $\gamma_d^a(T')$-set. Obviously, $D' \cup \{v, t\}$ is a DOIDS of the tree $T$. Thus $\gamma_d^a(T) \leq \gamma_d^a(T') + 2$. We now get $\gamma_d^a(T) \leq \gamma_d^a(T') + 2 \leq (2n' + l' + s')/3 + 2 = (2n - 4 + l - 1 + s - 1)/3 + 2 = (2n + l + s)/3$. If $\gamma_d^a(T) = (2n + l + s)/3$, then $\gamma_d^a(T') = (2n' + l' + s')/3$. By the inductive hypothesis we have $T' \in \mathcal{T}$. The tree $T$ can be obtained from $T'$ by operation $O_3$. Thus $T \in \mathcal{T}$.

Now assume that some child of $u$, say $x$, is a leaf. Let $T' = T - t$. We have $n' = n - 1$, $l' = l$ and $s' = s - 1$. Let $D'$ be any $\gamma_d^a(T')$-set. By Observation 1 we have $v \in D'$. It is easy to see that $D' \cup \{t\}$ is a DOIDS of the tree $T$. Thus $\gamma_d^a(T) \leq \gamma_d^a(T') + 1$. We now get $\gamma_d^a(T) \leq \gamma_d^a(T') + 1 \leq (2n' + l' + s')/3 + 1 = (2n - 2 + l + s - 1)/3 + 1 = (2n + l + s)/3$. If $\gamma_d^a(T) = (2n + l + s)/3$, then $\gamma_d^a(T') = (2n' + l' + s')/3$. By the inductive hypothesis we have $T' \in \mathcal{T}$. The tree $T$ can be obtained from $T'$ by operation $O_2$. Thus $T \in \mathcal{T}$.

If $d_T(u) = 1$, then $T = P_3 = T_1 \in \mathcal{T}$. By Lemma 3 we have $\gamma_d^a(T) = (2n + l + s)/3$. Now consider the case when $d_T(u) = 2$. First assume that there is a child of $w$ other than $u$, say $k$, such that the distance of $w$ to the most distant vertex of $T_k$ is three. It suffices to consider only the possibility when $T_k$ is a path $P_3$. Let $T' = T - T_u$. We have $n' = n - 3$, $l' = l - 1$ and $s' = s - 1$. Let us observe that there exists a $\gamma_d^a(T')$-set that does not contain the vertex $k$. Let $D'$ be such a set. The set $V(T') \setminus D'$ is independent, thus $w \in D'$. It is easy to observe that $D' \cup \{v, t\}$ is a DOIDS of the tree $T$. Thus $\gamma_d^a(T) \leq \gamma_d^a(T') + 2$. We now get $\gamma_d^a(T) \leq \gamma_d^a(T') + 2 \leq (2n' + l' + s')/3 + 2 = (2n - 6 + l - 1 + s - 1)/3 + 2 = (2n + l + s)/3 - 2/3 < (2n + l + s)/3$.

Now suppose that $w$ is adjacent to a leaf. Let $T' = T - T_u$. We have $n' = n - 3$, $l' = l - 1$ and $s' = s - 1$. Let $D'$ be any $\gamma_d^a(T')$-set. By Observation 2 we have $w \in D'$. It is easy to observe that $D' \cup \{v, t\}$ is a DOIDS of the tree $T$. Thus $\gamma_d^a(T) \leq \gamma_d^a(T') + 2$. We now get $\gamma_d^a(T) \leq \gamma_d^a(T') + 2 \leq (2n' + l' + s')/3 + 2 = (2n - 6 + l - 1 + s - 1)/3 + 2 = (2n + l + s)/3 - 2/3 < (2n + l + s)/3$. Henceforth, we can assume that $w$ is not adjacent to any leaf.
Now suppose that there is a child of $w$, say $k$, such that the distance of $w$ to the most distant vertex of $T_k$ is two. It suffices to consider only the possibility when $k$ is a support vertex of degree two. The leaf adjacent to $k$ we denote by $l$. Let $T' = T - T_u - l$. We have $n' = n - 4$, $l' = l - 1$ and $s' = s - 1$. Let $D'$ be any $\gamma_d(T')$-set. By Observations 1 and 2 we have $k, w \in D'$. It is easy to observe that $D' \cup \{v, t, l\}$ is a DOIDS of the tree $T$. Thus $\gamma_d(T) \leq \gamma_d(T') + 3$. We now get $\gamma_d(T) \leq \gamma_d(T') + 3 \leq (2n' + l' + s')/3 + 3 = (2n - 8 + l - 1 + s - 1)/3 + 3 = (2n + l + s)/3 - 1/3 < (2n + l + s)/3$.

If $d_T(w) = 1$, then $T = P_4$. We have $T \in \mathcal{T}$ as it can be obtained from $P_3$ by operation $O_2$. By Lemma 3 we have $\gamma_d(T) = (2n + l + s)/3$. Now consider the case when $d_T(w) = 2$. Let $T' = T - T_u$. Let $D'$ be any $\gamma_d(T')$-set. By Observation 1 we have $w \in D'$. It is easy to see that $D' \cup \{v, t\}$ is a DOIDS of the tree $T$. Thus $\gamma_d(T) \leq \gamma_d(T') + 2$. First suppose that $d$ is adjacent to a leaf. We have $n' = n - 3$, $l' = l$ and $s' = s - 1$. We now get $\gamma_d(T) \leq \gamma_d(T') + 2 \leq (2n' + l' + s')/3 + 2 = (2n - 6 + l + s - 1)/3 + 2 = (2n + l + s)/3 - 1/3 < (2n + l + s)/3$.

Now assume that no neighbor of $d$ is a leaf. Let $T' = T - T_u$. We have $n' = n - 3$, $l' = l$ and $s' = s$. We now get $\gamma_d(T) \leq \gamma_d(T') + 2 \leq (2n' + l' + s')/3 + 2 = (2n - 6 + l + s)/3 + 2 = (2n + l + s)/3$. If $\gamma_d(T) = (2n + l + s)/3$, then $\gamma_d(T') = (2n' + l' + s')/3$. By the inductive hypothesis we have $T' \in \mathcal{T}$. The tree $T$ can be obtained from $T'$ by operation $O_4$. Thus $T \in \mathcal{T}$.

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