HAT PROBLEM ON ODD CYCLES

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Abstract. The topic is the hat problem in which each of n players is randomly fitted with a blue or red hat. Then everybody can try to guess simultaneously his own hat color by looking at the hat colors of the other players. The team wins if at least one player guesses his hat color correctly, and no one guesses his hat color wrong; otherwise the team loses. The aim is to maximize the probability of a win. In this version every player can see everybody excluding himself. We consider such a problem on a graph, where vertices correspond to players, and a player can see each player to whom he is connected by an edge. The hat problem on a graph was solved for trees and for the cycle on four vertices. Then Uriel Feige conjectured that for any graph the maximum chance of success in the hat problem is equal to the maximum chance of success for the hat problem on the maximum clique in the graph. He provided several results that support this conjecture, and solved the hat problem for bipartite graphs and planar graphs containing a triangle. We make a step towards proving the conjecture of Feige. We solve the hat problem on all cycles of odd length. Of course, the maximum chance of success for the hat problem on the cycle on three vertices is three fourths. We prove that the hat number of every odd cycle of length at least five is one half, which is consistent with the conjecture of Feige.

1. Introduction

In the hat problem, a team of n players enters a room and a blue or red hat is randomly placed on the head of each player. Each player can see the hats of all of the other players but not his own. No communication of any sort is allowed, except for an initial strategy session before the game begins. Once they have had a chance to look at the other hats, each player must simultaneously guess the color of his own hat or pass. The team wins if at least one player guesses his hat color correctly and no one guesses his hat color wrong; otherwise the team loses. The aim is to maximize the probability of a win.

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The hat problem with seven people called "seven prisoners puzzle" was formulated by T. Ebert in his Ph.D. Thesis [4]. There are known many variations of the hat problem (for a comprehensive list, see [9]). For example in [6] there was considered a variation in which players do not have to guess their hat colors simultaneously. In [2] there was considered a variation in which passing is not allowed, thus everybody has to guess his hat color. The aim is to maximize the number of correct guesses. N. Alon [1] has proved a lower bound on the chance of success for the generalized hat problem with n people and q colors. This problem was also studied in [10]. The hat problem with three people was the subject of an article in The New York Times [11].

We consider the hat problem on a graph, where vertices correspond to players and a player can see each player to whom he is connected by an edge. This variation of the hat problem was first considered in [7]. There were proven some general theorems about the hat problem on a graph, and the problem was solved on trees. Additionally, there was considered the hat problem on a graph such that the only known information are degrees of vertices. In [8] the problem was solved on the cycle C_4. Uriel Feige [5] conjectured that for any graph the maximum chance of success in the hat problem is equal to the maximum chance of success for the hat problem on the maximum clique in the graph. He provided several results that support this conjecture, and solved the hat problem for bipartite graphs and planar graphs containing a triangle. Feige proved that if a graph is such that the chromatic number equals the number of vertices of the maximum clique, then the conjecture is true. A well known class of graphs for which the chromatic number equals the number of vertices of the maximum clique is that of perfect graphs (where that equality holds not only for the graph, but also for all its subgraphs). Thus Feige solved the hat problem for all perfect graphs. By the strong perfect graph theorem [3], every graph for which neither it nor its complement contains an induced odd cycle of length at least five is perfect. We solve the hat problem on all cycles of odd length. Of course, the maximum chance of success for the hat problem on the cycle on three vertices is three fourths. We prove that the hat number of every odd cycle of length at least five is one half, which is consistent with the conjecture of Feige.

2. Preliminaries

For a graph G, the set of vertices and the set of edges we denote by $V(G)$ and $E(G)$, respectively. If H is a subgraph of G, then we write $H \subseteq G$. The path (cycle, respectively) on n vertices we denote by P_n (C_n, respectively). The neighborhood of a vertex v of G, that is $\{x \in V(G)\colon vx \in E(G)\}$, we denote by $N_G(v)$. We say that a vertex v is neighborhood-dominated if there is some other vertex u such that $N_G(v) \subseteq N_G(u)$.

Let $f\colon X \to Y$ be a function. If for every $x \in X$ we have $f(x) = y$, then we write $f \equiv y$.

Let $V(G) = \{v_1, v_2, \ldots, v_n\}$. By $Sc = \{1, 2\}$ we denote the set of colors, where 1 corresponds to the blue color, and 2 corresponds to the red color.

By a case for a graph G we mean a function $c\colon V(G) \to \{1, 2\}$, where $c(v_i)$ means color of vertex v_i. The set of all cases for the graph G we denote by $C(G)$, of course $|C(G)| = 2^{|V(G)|}$. If $c \in C(G)$, then to simplify notation, we write $c = c(v_1)c(v_2)\ldots c(v_n)$ instead of $c = \{(v_1, c(v_1)), (v_2, c(v_2)), \ldots, (v_n, c(v_n))\}$. For example, if a case $c \in C(C_5)$ is such that $c(v_1) = 2$, $c(v_2) = 1$, $c(v_3) = 1$, $c(v_4) = 2$, and $c(v_5) = 1$, then we write $c = 21121$.

By a situation of a vertex v_i we mean a function $s_i\colon V(G) \to Sc \cup \{0\} = \{0, 1, 2\}$, where $s_i(v_j) \in Sc$ if v_i and v_j are adjacent, and 0 otherwise. The set of all possible situations of v_i in the graph G we denote by $St_i(G)$, of course $|St_i(G)| = 2^{d_G(v_i)}$. If $s_i \in St_i(G)$, then for simplicity of notation, we write $s_i = s_i(v_1)s_i(v_2)\ldots s_i(v_n)$ instead of $s_i = \{(v_1, s_i(v_1)), (v_2, s_i(v_2)), \ldots, (v_n, s_i(v_n))\}$. For example, if $s_3 \in St_3(C_5)$ is such that $s_3(v_2) = 2$ and $s_3(v_4) = 1$, then we write $s_3 = 02010$.

By a guessing instruction of a vertex $v_i \in V(G)$ we mean a function $g_i\colon St_i(G) \to Sc \cup \{0\} = \{0, 1, 2\}$, which for a given situation gives the color v_i guesses it is, or 0 if v_i passes. Thus guessing instruction is a rule determining behavior of a vertex in every situation. We say that v_i never guesses its color if v_i passes in every situation, that is, $g_i \equiv 0$. We say that v_i always guesses its color if v_i guesses its color in every situation, that is, for every $s_i \in St_i(G)$ we have $g_i(s_i) \in \{1, 2\}$ ($g_i(s_i) \neq 0$, equivalently).

Let c be a case, and let s_i be the situation (of vertex v_i) corresponding to that case. The guess of v_i in the case c is correct (wrong, respectively) if $g_i(s_i) = c(v_i)$ ($0 \neq g_i(s_i) \neq c(v_i)$, respectively). By result of the case c we mean a win if at least one vertex guesses its color correctly, and no vertex guesses its color wrong, that is, $g_i(s_i) = c(v_i)$ (for some i) and there is no j such that $0 \neq g_j(s_j) \neq c(v_j)$. Otherwise the result of the case c is a loss.

By a strategy for the graph G we mean a sequence $(g_1, g_2, \ldots, g_n)$, where g_i is the guessing instruction of vertex v_i. The family of all strategies for a graph G we denote by $\mathcal{F}(G)$.

If $S \in \mathcal{F}(G)$, then the set of cases for the graph G for which the team wins (loses, respectively) using the strategy S we denote by $W(S)$ ($L(S)$, respectively). By the chance of success of the strategy S we mean the number $p(S) = |W(S)|/|C(G)|$. By the hat number of the graph G we mean the number $h(G) = \max\{p(S)\colon S \in \mathcal{F}(G)\}$. We say that a strategy S is optimal for the graph G if $p(S) = h(G)$. The family of all optimal strategies for the graph G we denote by $\mathcal{F}^0(G)$.

By solving the hat problem on a graph G we mean finding the number $h(G)$.

The following four results are from [7].

Theorem 2.1. *If H is a subgraph of G, then $h(H) \leq h(G)$.*

Corollary 2.2. *For every graph G we have $h(G) \geq 1/2$.*

The following theorem is the solution of the hat problem on paths.

Theorem 2.3. *For every path P_n we have $h(P_n) = 1/2$.*

Now there is a sufficient condition for the removal of a vertex of a graph without changing its hat number.

Theorem 2.4. *Let G be a graph, and let v be a vertex of G. If there exists a strategy $S \in \mathcal{F}^0(G)$ such that v never guesses its color, then $h(G) = h(G - v)$.*

Uriel Feige [5] proved the following result.

Lemma 2.5. *Let G be a graph. If v is a neighborhood-dominated vertex of G, then $h(G) = h(G - v)$.*

3. Results

To solve the hat problem on odd cycles of length at least five, we need the fact that $h(C_5) = 1/2$, see Lemma 3.2. Now we prove our main result.

Theorem 3.1. *If $n \geq 5$ is an odd integer, then $h(C_n) = 1/2$.*

Proof. We obtain the result by induction on the length of the cycle. For $n = 5$ the theorem is true by Lemma 3.2. Now assume that $n \geq 7$ is an odd integer, and $h(C_{n-2}) = 1/2$. Let $H_n = C_n \cup v_1v_4$. By Theorem 2.1 we have $h(H_n) \geq h(C_n)$. Observe that $N_{H_n}(v_3) \subset N_{H_n}(v_1)$. Let $H'_n = H_n - v_3$. By Lemma 2.5 we get $h(H_n) = h(H'_n)$. Moreover, since $N_{H'_n}(v_2) \subset N_{H'_n}(v_n)$, again by Lemma 2.5 we get $h(H'_n) = h(H'_n - v_2)$. Let us observe that the graph $H'_n - v_2$ is isomorphic to the cycle C_{n-2}. By the inductive hypothesis we have $h(C_{n-2}) = 1/2$. Now we get $h(C_n) \leq h(H_n) = h(H'_n) = h(C_{n-2}) = 1/2$. On the other hand, by Corollary 2.2 we have $h(C_n) \geq 1/2$. $\square$

Now we solve the hat problem on the cycle on five vertices.

Lemma 3.2. $h(C_5) = 1/2$.

Proof. Let S be an optimal strategy for C_5. If some vertex, say v_i, never guesses its color, then by Theorem 2.4 we have $h(C_5) = h(C_5 - v_i)$. Since $C_5 - v_i = P_4$ and $h(P_4) = 1/2$ (by Theorem 2.3), we get $h(C_5) = 1/2$. Now assume that every vertex guesses its color.

Let us consider a guessing instruction of a vertex. If in every case in which this instruction gives a correct guess some other vertex also guesses its color, then we say that the guessing instruction is dominated. Let us observe that we do not have to consider strategies with a dominated guessing instruction because such instruction cannot improve the chance of success. Even if it is the only one guess of a vertex, then by Theorem 2.4 we get $p(S) \leq h(C_5 - v_i) = h(P_4) = 1/2$ implying that $h(C_5) = 1/2$.

Now we explain a way in which the result can be easily verified using computer. We consider only guessing instructions which are not passing. First consider strategies S with exactly one instruction for every vertex. There are exactly 8 possible instructions for each vertex (because of the colors of two neighbors and the guess it is going to make). Thus the total number of possibilities for S is $8^5 = 2^{15}$. Let us observe that from a strategy we can obtain a group of 320 (not necessarily distinguishable) symmetrical strategies. We can perform each one of the following operations: rotating the vertices (gives 5 possibilities), reflecting the vertices (gives 2 possibilities), and relabelling the colors of the vertices (gives $2^5 = 32$ possibilities). Reducing modulo this symmetry group gives only 120 possibilities for S. Now for every one of these possibilities we check the number of cases in which some vertex guesses its color wrong. If in at least 16 cases some vertex guesses its color wrong, then the team loses for at least 16 cases implying that $p(S) \leq 1/2$. For 61 of those 120 strategies in at least 16 cases some vertex guesses its color wrong. Therefore it suffices to consider only the remaining 59 strategies. Now we reduce the set of possibilities by using the idea of dominance from the previous paragraph. In this way we exclude 37 strategies, having only 22 strategies left. Now for every one of them we check the number of cases in which the team so far wins, that is, cases in which some vertex guesses its color correctly while at the same time no vertex guesses its color wrong. The best score among those 22 strategies is 12 successes. Thus we now examine adding an additional instruction to each one of the 22 strategies. Again we exclude strategies for which the team loses for at least 16 cases, or some guessing instruction is dominated. As a result there are only 23 strategies (each one consisting of six guessing instructions) left. Among them, the best score of cases for which the team wins is 12. Now we try to add an additional instruction to each one of the 23 strategies. We verify that for every one of them the team loses for at least 16 cases, or some guessing instruction is dominated. This implies that for every strategy $S \in \mathcal{F}(C_5)$ we have $p(S) \leq 1/2$. Now, by definition we get $h(C_5) = 1/2$. $\square$

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