2-bondage in graphs

Marcin Krzywkowski*
e-mail: marcin.krzywkowski@gmail.com

Department of Algorithms and System Modelling
Faculty of Electronics, Telecommunications and Informatics
Gdańsk University of Technology
Narutowićza 11/12, 80–233 Gdańsk, Poland
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A 2-dominating set of a graph \( G = (V, E) \) is a set \( D \) of vertices of \( G \) such that every vertex of \( V(G) \setminus D \) has at least two neighbors in \( D \). The 2-domination number of a graph \( G \), denoted by \( \gamma_2(G) \), is the minimum cardinality of a 2-dominating set of \( G \). The 2-bondage number of \( G \), denoted by \( b_2(G) \), is the minimum cardinality among all sets of edges \( E' \subseteq E \) such that \( \gamma_2(G - E') > \gamma_2(G) \). If for every \( E' \subseteq E \) we have \( \gamma_2(G - E') = \gamma_2(G) \), then we define \( b_2(G) = 0 \), and we say that \( G \) is a \( \gamma_2 \)-strongly stable graph. First we discuss the basic properties of 2-bondage in graphs. We find the 2-bondage numbers for several classes of graphs. Next we show that for every non-negative integer there exists a tree with such 2-bondage number. Finally, we characterize all trees with 2-bondage number equaling one or two.

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1. Introduction

Let \( G = (V, E) \) be a graph. By the neighborhood of a vertex \( v \) of \( G \) we mean the set \( N_G(v) = \{ u \in V(G) : uv \in E(G) \} \). The degree of a vertex \( v \), denoted by \( d_G(v) \), is the cardinality of its neighborhood. By a leaf we mean a vertex of degree one, while a support vertex is a vertex adjacent to a leaf. We say that a support vertex is strong if it is adjacent to at least two leaves. The distance between two vertices of a graph is the number of edges in a shortest path connecting them. The eccentricity of a vertex is the greatest distance between it and any other vertex. The diameter of a graph \( G \), denoted by \( \text{diam}(G) \), is the maximum eccentricity among all vertices of \( G \). The path (cycle, respectively) on \( n \) vertices is denoted by \( P_n \) (\( C_n \), respectively). A wheel \( W_n \), where \( n \geq 4 \), is a graph with \( n \) vertices, formed by connecting a vertex to all vertices of the cycle \( C_{n-1} \). By a star we mean a connected graph in which exactly one vertex has degree greater than one. Let \( K_{p,q} \) denote a complete bipartite graph the partite sets of which have cardinalities \( p \) and \( q \).

A subset \( D \subseteq V(G) \) is a dominating set of \( G \) if every vertex of \( V(G) \setminus D \) has a neighbor in \( D \), while it is a 2-dominating set, abbreviated as 2DS, of \( G \) if every vertex of \( V(G) \setminus D \) has at least two neighbors in \( D \). The domination (2-domination, respectively) number of a graph \( G \), denoted by \( \gamma(G) \) (\( \gamma_2(G) \), respectively), is the minimum cardinality of a dominating (2-dominating, respectively) set of \( G \). Note that 2-domination is a type of multiple domination in which each vertex, which is

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not in the dominating set, is dominated at least \( k \) times for a fixed positive integer \( k \). Multiple domination was introduced by Fink and Jacobson [3], and further studied for example in [1, 13]. For a comprehensive survey of domination in graphs, see [7, 8].

The bondage number \( b(G) \) of a graph \( G \) is the minimum cardinality among all sets of edges \( E' \subseteq E \) such that \( \gamma(G - E') > \gamma(G) \). If for every \( E' \subseteq E \) we have \( \gamma(G - E') = \gamma(G) \), then we define \( b(G) = 0 \), and we say that \( G \) is a \( \gamma \)-strongly stable graph. Bondage in graphs was introduced in [4], and further studied for example in [2, 5, 6, 9–12, 14].

We define the 2-bondage number of \( G \), denoted by \( b_2(G) \), to be the minimum cardinality among all sets of edges \( E' \subseteq E \) such that \( 2(\gamma(G - E')) > 2(\gamma(G)) \). Thus \( b_2(G) \) is the minimum number of edges of \( G \) that have to be removed in order to increase the 2-domination number. If for every \( E' \subseteq E \) we have \( 2(\gamma(G - E')) = 2(\gamma(G)) \), then we define \( b_2(G) = 0 \), and we say that \( G \) is a 2-strongly stable graph.

First we discuss the basic properties of 2-bondage in graphs. We find the 2-bondage numbers for several classes of graphs. Next we show that for every non-negative integer there exists a tree with such 2-bondage number. Finally, we characterize all trees with 2-bondage number equaling one or two.

2. Results

We begin with the following observations.

Observation 2 Every leaf of a graph \( G \) is in every \( \gamma_2(G) \)-set.

Observation 3 If \( H \subseteq G \) and \( V(H) = V(G) \), then \( \gamma_2(H) \geq \gamma_2(G) \).

Observation 4 For every positive integer \( n \) we have \( \gamma_2(K_n) = \min \{2, n\} \).

Observation 5 If \( n \) is a positive integer, then \( \gamma_2(P_n) = \lfloor n/2 \rfloor + 1 \).

Observation 6 For every integer \( n \geq 3 \) we have \( \gamma_2(C_n) = \lfloor (n + 1)/2 \rfloor \).

Observation 7 For every integer \( n \geq 4 \) we have

\[
\gamma_2(W_n) = \begin{cases} 
2 & \text{if } n = 4, 5; \\
\lfloor (n + 1)/3 \rfloor + 1 & \text{if } n \geq 6.
\end{cases}
\]

Observation 8 Let \( p \) and \( q \) be positive integers such that \( p \leq q \). Then

\[
\gamma_2(K_{p,q}) = \begin{cases} 
\max \{q, 2\} & \text{if } p = 1; \\
\min \{p, 4\} & \text{if } p \geq 2.
\end{cases}
\]

First we find the 2-bondage numbers of complete graphs.

Proposition 9 For every positive integer \( n \) we have

\[
b_2(K_n) = \begin{cases} 
0 & \text{if } n = 1, 2; \\
\lfloor 2n/3 \rfloor & \text{otherwise}.
\end{cases}
\]

Proof Obviously, \( b_2(K_1) = 0 \) and \( b_2(K_2) = 0 \). Now assume that \( n \geq 3 \). Let \( V(K_n) = \{v_1, v_2, \ldots, v_n\} \). Observe that the 2-domination number of a graph equals two if and only if there is a pair of vertices, which are both adjacent to all vertices other than themselves. Let \( E' \subseteq E(K_n) \). Let us observe \( \gamma_2(K_n - E') > 2 \) if and only if at most one vertex of \( K_n \) is not incident to any edge of \( E' \), and
every edge of $E'$ is adjacent to some other edge of $E'$. We want to choose a smallest set $E' \subseteq E(K_n)$ satisfying the above condition. Let us observe that the most efficient way is to choose for example the edges $v_1v_2, v_2v_3, v_4v_5, v_5v_6$, and so on. Let $k$ be a positive integer. If $n = 3k$, then we have to remove $2k$ edges. Thus $b_2(K_{3k}) = 2k = 2n/3 = [2n/3]$. If $n = 3k + 1$, then we also remove $2k$ edges as one vertex can remain universal. We have $b_2(K_{3k+1}) = 2k = [2k + 2]/3 = [2(3k + 1)/3] = [2n/3]$. Now assume that $n = 3k + 2$. If we remove the edges $v_1v_2, v_2v_3, v_4v_5, v_5v_6, \ldots, v_{3k-2}v_{3k-1}, v_{3k-1}v_{3k}$, then the vertices $v_{3k+1}$ and $v_{3k+2}$ remain universal. Therefore $b_2(K_{3k+2}) > 2k$. Let us observe that removing also the edge $v_{3k}v_{3k+1}$ suffices to increase the 2-domination number. This implies that $b_2(K_{3k+2}) = 2k + 1 = [2k + 4/3] = [2(3k + 2)/3] = [2n/3]$.

Now we calculate the 2-bondage numbers of paths.

**Proposition 10** If $n$ is a positive integer, then

$$b_2(P_n) = \begin{cases} 
0 & \text{for } n = 1, 2; \\
1 & \text{for } n \geq 3.
\end{cases}$$

Now we investigate the 2-bondage in cycles.

**Proposition 11** For every integer $n \geq 3$ we have

$$b_2(C_n) = \begin{cases} 
1 & \text{if } n \text{ is even;} \\
2 & \text{if } n \text{ is odd.}
\end{cases}$$

Now we calculate the 2-bondage numbers of wheels.

**Proposition 12** For every integer $n \geq 4$ we have

$$b_2(W_n) = \begin{cases} 
1 & \text{if } n = 5; \\
2 & \text{if } n \neq 3k + 2; \\
3 & \text{otherwise.}
\end{cases}$$

**Proof** Let $E(W_n) = \{v_1v_2, v_1v_3, \ldots, v_1v_n, v_2v_3, v_3v_4, \ldots, v_{n-1}v_n, v_nv_2\}$. Using Proposition 9 we get $b_2(W_3) = b_2(K_3) = 2$. By Observation 7 we have $\gamma_2(W_5) = 2$. We also have $\gamma_2(W_5 - v_2v_3) = 3 > 2 = \gamma_2(W_5)$. Thus $b_2(W_5) = 1$. Now assume that $n \geq 6$. If we remove an edge incident to $v_1$, say $v_1v_2$, then we get $\gamma_2(W_n - v_1v_2) = \gamma_2(W_n)$ as we can construct a $\gamma_2(W_n)$-set that contains the vertices $v_1$ and $v_2$; such set is also a 2DS of the graph $W_n - v_1v_2$. If we remove an edge non-incident to $v_1$, say $v_2v_3$, then we get $\gamma_2(W_n - v_2v_3) = \gamma_2(W_n)$ as we can construct a $\gamma_2(W_n)$-set that does not contain the vertices $v_2$ and $v_3$; such set is also a 2DS of the graph $W_n - v_2v_3$. This implies that $b_2(W_n) \neq 1$. First assume that $n = 3k$ or $n = 3k + 1$. Let us remove two edges non-incident to $v_1$ and incident to the same vertex $v_i$ (for some $i \neq 1$). For example, we remove the edges $v_{n-1}v_n$ and $v_nv_2$. Now we find a relation between the numbers $\gamma_2(W_n - v_{n-1}v_n - v_nv_2)$ and $\gamma_2(W_n - v_n)$. Let $D$ be any $\gamma_2(W_n - v_{n-1}v_n - v_nv_2)$-set. By Observation 2 we have $v_n \in D$. Let us observe that $D \setminus \{v_n\}$ is a 2DS of the graph $W_n - v_n$. Thus $\gamma_2(W_n - v_n) \leq \gamma_2(W_n - v_{n-1}v_n - v_nv_2) - 1$. Observe that $W_n - v_n$ is a subgraph of $W_{n-1}$ having the same set of vertices, as $W_{n-1} - v_{n-1}v_2 = W_n - v_n$. Using Observations 3 and 7 we get $\gamma_2(W_n - v_{n-1}v_n - v_nv_2) \geq \gamma_2(W_n - v_n) + 1 \geq \gamma_2(W_{n-1}) + 1 = \lceil n/3 \rceil + 2 = \lceil (n + 1)/3 \rceil + 2 = \gamma_2(W_n) + 1 > \gamma_2(W_n)$. Therefore $b_2(W_n) = 2$ if $n = 3k$ or $n = 3k + 1$. Now assume that $n = 3k + 2$. It is not very difficult to verify that now removing any two edges does not increase the 2-domination number. This implies that $b_2(W_n) \neq 1, 2$. Let us re-
move three edges non-incident to $v_1$, and forming a path $P_4$. For example, we remove the edges $v_{n-2}v_{n-1}$, $v_{n-1}v_n$, and $v_nv_2$. Now we find a relation between the numbers $\gamma_2(W_n - v_{n-2}v_{n-1} - v_{n-1}v_n - v_nv_2)$ and $\gamma_2(W_n - v_{n-1} - v_n)$. Let $D$ be any $\gamma_2(W_n - v_{n-2}v_{n-1} - v_{n-1}v_n - v_nv_2)-$set. By Observation 2 we have $v_{n-1}, v_n \in D$. Let us observe that $D \setminus \{v_{n-1}, v_n\}$ is a 2DS of the graph $W_n - v_{n-1} - v_n$. Thus $\gamma_2(W_n - v_{n-1} - v_n) \leq \gamma_2(W_n - v_{n-2}v_{n-1} - v_{n-1}v_n - v_nv_2) - 2$. Observe that $W_n - v_{n-1} - v_n$ is a subgraph of $W_{n-2}$ having the same set of vertices, as $W_{n-2} - v_{n-2}v_{n-1} = W_n - v_{n-1} - v_n$. Using Observations 3 and 7 we get $\gamma_2(W_n - v_{n-2}v_{n-1} - v_{n-1}v_n - v_nv_2) \geq \gamma_2(W_n - v_{n-1} - v_n) + 2 \geq \gamma_2(W_{n-2}) + 2 = \left\lfloor \frac{(n - 1)}{3} \right\rfloor + 3 = \left\lfloor \frac{(3k + 1)}{3} \right\rfloor + 3 = \left\lfloor \frac{(3k + 3)}{3} \right\rfloor + 2 = \left\lfloor \frac{(n + 1)}{3} \right\rfloor + 2 = \gamma_2(W_n) + 1 > \gamma_2(W_n)$. Therefore $b_2(W_n) = 3$ if $n = 3k + 2$.

Now we investigate the 2-bondage in complete bipartite graphs.

**Proposition 13** Let $p$ and $q$ be positive integers such that $p \leq q$. Then

$$b_2(K_{p,q}) = \begin{cases} q - 1 & \text{if } p = 1; \\ 3 & \text{if } p = q = 3; \\ 5 & \text{if } p = q = 4; \\ p - 1 & \text{otherwise.} \end{cases}$$

**Proof** Let $E(K_{p,q}) = \{a_ib_j : 1 \leq i \leq p \text{ and } 1 \leq j \leq q\}$. If $p = 1$, then $K_{p,q}$ is a star. We have $b_2(K_{1,1}) = 0 = q - 1$. If $q \geq 2$, then it is not difficult to verify that in order to increase the 2-domination number we have to remove all but one edge of $K_{1,q}$. Thus $b_2(K_{1,q}) = q - 1$.

Now assume that $p = 2$. By Observation 8 we have $\gamma_2(K_{2,q}) = 2$. Let us observe that $\gamma_2(K_{2,q} - a_1b_1) = 3$. Consequently, $b_2(K_{2,q}) = 1 = p - 1$.

Now let us assume that $p = 3$. By Observation 8 we have $\gamma_2(K_{3,q}) = 3$. If $q = 3$, then it is not difficult to verify that removing any two edges does not increase the 2-domination number. We have $\gamma_2(K_{3,3} - a_1b_1 - a_1b_2 - a_2b_1) = 4 > 3 = \gamma_2(K_{3,3})$. Therefore $b_2(K_{3,3}) = 3$. Now assume that $q \geq 4$. It is easy to see that removing one edge does not increase the 2-domination number. Let us observe that $\gamma_2(K_{3,q} - a_1b_1 - a_2b_1) = 4$. Therefore $b_2(K_{3,q}) = 2 = p - 1$ if $q \geq 4$.

Now assume that $p \geq 4$. By Observation 8 we have $\gamma_2(K_{p,q}) = 4$. If $q = 4$, then it is not very difficult to verify that removing any four edges does not increase the 2-domination number. Let us observe that $\gamma_2(K_{4,4} - a_1b_1 - a_1b_2 - a_1b_3 - a_2b_1 - a_3b_1) = 5$. Consequently, $b_2(K_{4,4}) = 5$. Now assume that $q \geq 5$. Let $E'$ be a subset of the set of edges of $K_{p,q}$, and let $H = K_{p,q} - E'$. Let us observe that if there are vertices $a_i$ and $a_j$ such that $d_H(a_i) = d_H(a_j) = q$ and vertices $b_k$ and $b_l$ such that $d_H(b_k) = d_H(b_l) = p$, then $b_2(H) = 4$. Therefore removing any $p - 2$ edges of $K_{p,q}$ does not increase the 2-domination number. Let $E' = \{a_1b_1, a_2b_1, \ldots, a_{p-1}b_1\}$. We have $\gamma_2(H) = 5$ as the vertex $b_1$ has to belong to every 2DS of the graph $H$. This implies that $b_2(K_{p,q}) = p - 1$ if $p \geq 4$ and $q \geq 5$.

A paired dominating set of a graph $G$ is a dominating set of vertices whose induced subgraph has a perfect matching. The paired domination number of $G$, denoted by $\gamma_p(G)$, is the minimum cardinality of a paired dominating set of $G$. The paired bondage number, denoted by $b_p(G)$, is the minimum cardinality among all sets of edges $E' \subseteq E$ such that $\delta(G - E') \geq 1$ and $\gamma_p(G - E') < \gamma_p(G)$. If for every $E' \subseteq E$, either $\gamma_p(G - E') = \gamma_p(G)$ or $\delta(G - E') = 0$, then we define $b_p(G) = 0$, and we say that $G$ is a $\gamma_p$-strongly stable graph. Raczek [11] noticed that if $H \subseteq G$, then $b_p(H) \leq b_p(G)$. Let us observe that no inequality of such type is possible for the 2-bondage. Consider the complete bipartite graphs $K_{1,3}$, $K_{2,3}$, and $K_{3,3}$. Obviously, $K_{1,3} \subseteq K_{2,3} \subseteq K_{3,3}$. Using Proposition 13 we get $b_2(K_{1,3})$.
The authors of [4] proved that the bondage number of any tree is either one or two. Let us observe that for any non-negative integer there exists a tree with such 2-bondage number, as by Proposition 13 we have \( b_2(K_{1,m}) = m - 1 \). Obviously, \( b_2(P_1) = 0 \) and \( b_2(P_2) = 0 \). Let us observe that the paths \( P_1 \) and \( P_2 \) are the only \( \gamma_2 \)-strongly stable trees. We characterize all trees with 2-bondage number equaling one or two.

Let \( T_0 \) be a family of trees that have a strong support vertex of degree three, or a vertex adjacent to at least two support vertices of degree two, or a vertex which does not belong to any minimum 2-dominating set and is adjacent to a star \( K_{1,3} \) through the central vertex.

Now we prove that the 2-bondage number of every tree of the family \( T_0 \) is either one or two.

**Lemma 14** If \( T \in T_0 \), then \( b_2(T) \in \{1, 2\} \).

**Proof** First assume that \( T \) has a strong support vertex, say \( x \), of degree three. Let \( y \) and \( z \) be leaves adjacent to \( x \). The neighbor of \( x \) other than \( y \) and \( z \) is denoted by \( t \). Let \( T' = T - x - y - z \). Let \( D' \) be any \( \gamma_2(T') \)-set. It is easy to observe that \( D' \cup \{y, z\} \) is a 2DS of the tree \( T \). Thus \( \gamma_2(T) \leq \gamma_2(T') + 2 \). Now we get \( \gamma_2(T - xy) = \gamma_2(T') + \gamma_2(P_1) + \gamma_2(P_2) = \gamma_2(T') + 3 \geq \gamma_2(T) + 1 > \gamma_2(T) \). This implies that \( 0 \neq b_2(T) \leq 2 \), that is, \( b_2(T) \in \{1, 2\} \).

Now assume that \( T \) has a vertex, say \( x \), adjacent to at least two support vertices of degree two. One of them let us denote by \( y \). The leaf adjacent to \( y \) is denoted by \( z \). Let \( T' = T - y - z \). Let us observe that there exists a \( \gamma_2(T') \)-set that contains the vertex \( x \). Let \( D' \) be such a set. It is easy to see that \( D' \cup \{z\} \) is a 2DS of the tree \( T \). Thus \( \gamma_2(T) \leq \gamma_2(T') + 1 \). Now we get \( \gamma_2(T - xy) = \gamma_2(T' \cup P_1 \cup P_2) = \gamma_2(T') + \gamma_2(P_1) + \gamma_2(P_2) = \gamma_2(T') + 2 \geq \gamma_2(T) + 1 > \gamma_2(T) \). This implies that \( b_2(T) = 1 \).

Now assume that \( T \) has a vertex, say \( x \), which does not belong to any \( \gamma_2(T) \)-set, and is adjacent to a star \( K_{1,3} \) through the central vertex, say \( y \). The leaves adjacent to \( y \) we denote by \( a, b, \) and \( c \). Let \( D \) be any \( \gamma_2(T) \)-set. By Observation 2 we have \( a, b, c \in D \). The vertex \( x \) does not belong to any \( \gamma_2(T) \)-set, thus \( x, y \notin D \). Let \( T' = T - a - b \). It is easy to observe \( D \setminus \{a, b\} \) is not a 2DS of the tree \( T' \) as the vertex \( y \) has only one neighbor in \( D \setminus \{a, b\} \). Therefore \( \gamma_2(T') > \gamma_2(T) - 2 \). Now we get \( \gamma_2(T - ya - yb) = \gamma_2(T' \cup P_1 \cup P_2) = \gamma_2(T') + 2 \gamma_2(P_1) = \gamma_2(T') + 2 > \gamma_2(T) \). This implies that \( b_2(T) \in \{1, 2\} \).\( \blacksquare \)

We characterize all trees with 2-bondage number equaling one or two. For this purpose we introduce a family \( T \), which consists of the path \( P_3 \), all trees of the family \( T_0 \), and trees \( T_k \) that can be obtained as follows. Let \( T_1 \) be an element of \( T_0 \). If \( k \) is a positive integer, then \( T_{k+1} \) can be obtained recursively from \( T_k \) by one of the following operations.

- **Operation \( O_1 \)**: Attach a star by joining the central vertex to any vertex of \( T_k \).
- **Operation \( O_2 \)**: Attach a path \( P_2 \) and a non-negative number of vertices to a leaf of \( T_k \).

Now we prove that the 2-bondage number of every tree of the family \( T \) is either one or two.

**Lemma 15** If \( T \in T \), then \( b_2(T) \in \{1, 2\} \).

**Proof** Obviously, \( b_2(P_3) = 1 \). If \( T \in T_0 \), then by Lemma 14 we have \( b_2(T) \in \{1, 2\} \). Now assume that \( T \in T \setminus (T_0 \cup \{P_3\}) \). We use the induction on the number \( k \) of operations performed to construct the tree \( T \). Let \( k \geq 2 \) be an integer. Assume that the result is true for every tree \( T' = T_k \) of the family \( T \) constructed by \( k - 1 \)
them from the set $D$ implies that all leaves adjacent to $x$. The vertex to which $x$ is attached is denoted by $y$. Let $D'$ be any $\gamma_2(T')$-set. It is easy to observe that the elements of the set $D'$ together with the leaves of the the attached star form a 2DS of the tree $T$. Thus $\gamma_2(T) \leq \gamma_2(T') + m$. The assumption $b_2(T') \in \{1, 2\}$ implies that there exists $E' \subseteq E(T')$ such that $|E'| \leq 2$ and $\gamma_2(T' - E') > \gamma_2(T')$. By $T'$ $(T''y$, respectively) we denote the component of $T - E'$ $(T' - E'$, respectively) which contains the vertex $y$. Let us observe that there exists a $\gamma_2(T')$-set that does not contain the vertex $x$. Let $D''$ be such a set. Observation 2 implies that all leaves of the attached star belong to the set $D''$. Observe that after removing the leaves of the attached star from the set $D''$ we get a 2DS of the tree $T''$. Therefore $\gamma_2(T'') \leq \gamma_2(T'') - m$. Now we get $\gamma_2(T - E') = \gamma_2(T - E' - T'') + \gamma_2(T'') \geq \gamma_2(T - E' - T'') + \gamma_2(T'') + m = \gamma_2(T' - E') + \gamma_2(T'') + m = \gamma_2(T' - E') + m > \gamma_2(T') + m \geq \gamma_2(T)$. This implies that $0 \neq b_2(T') \leq 2$, and consequently, $b_2(T') \in \{1, 2\}$.

Now assume that $T$ is obtained from $T'$ by Operation $O_2$. Assume that we attach one path $P_2$ and $k \geq 0$ vertices. The vertex to which are attached new vertices we denote by $x$. Let $D'$ be any $\gamma_2(T')$-set. By Observation 2 we have $x \in D'$. It is easy to observe that the elements of the set $D'$ together with all leaves of $T$ which do not exist in $T'$ form a 2DS of the tree $T$. Thus $\gamma_2(T) \leq \gamma_2(T') + k + 1$. The assumption $b_2(T') \in \{1, 2\}$ implies that there exists $E' \subseteq E(T')$ such that $|E'| \leq 2$ and $\gamma_2(T' - E') > \gamma_3(T')$. By $T'$ $(T''z$, respectively) we denote the component of $T - E'$ $(T' - E'$, respectively) which contains the vertex $x$. Let us observe that there exists a $\gamma_2(T'')$-set that contains the vertex $x$. Let $D''$ be such a set. Observation 2 implies that all leaves of $T$ which do not exist in $T'$ belong to the set $D''$. The set $D''$ is minimal, thus no vertex of $T$, which neither exists in the tree $T'$ nor is a leaf, belongs to the set $D''$. It is easy to observe that after removing from $D$ all leaves of $T$ which do not exist in $T'$ we get a 2DS of the tree $T''$. Therefore $\gamma_3(T''z) \leq \gamma_2(T'') - k - 1$. Now we get $\gamma_2(T - E') = \gamma_3(T - E' - T''z) + \gamma_2(T''z) \geq \gamma_2(T - E' - T''z) + \gamma_2(T''z) + k + 1 = \gamma_2(T' - E') + k + 1 \geq \gamma_2(T)$. This implies that $b_2(T') \in \{1, 2\}$.  

Now we prove that if the 2-bondage number of a tree equals one or two, then the tree belongs to the family $T$.

Lemma 16 Let $T$ be a tree. If $b_2(T) \in \{1, 2\}$, then $T \in T$.

Proof Let $n$ be the number of vertices of the tree $T$. We proceed by induction on this number. If $\text{diam}(T) \in \{0, 1\}$, then $T \in \{P_1, P_2\}$. We have $b_2(P_1) = b_2(P_2) = 0 \notin \{1, 2\}$. Now assume that $\text{diam}(T) = 2$. Thus $T$ is a star $K_{1,m}$. By Proposition 13 we have $b_2(K_{1,m}) = m - 1$. If $b_2(K_{1,m}) = 1$, then $m = 2$. We have $T = K_{1,2} = P_3 \in T$. If $b_2(K_{1,m}) = 2$, then $m = 3$. We have $T = K_{1,3} \in T_0 \subseteq T$ as $K_{1,3}$ has a strong support vertex of degree three.

Now assume that $\text{diam}(T) \geq 3$. Thus the order $n$ of the tree $T$ is at least four. We obtain the result by the induction on the number $n$. Assume that the lemma is true for every tree $T'$ of order $n' < n$. We root $T$ at a vertex $r$ of maximum eccentricity $\text{diam}(T)$. Let $t$ be a leaf at maximum distance from $r$, $v$ be the parent of $t$, and $u$ be the parent of $v$ in the rooted tree. If $\text{diam}(T) \geq 4$, then let $w$ be the parent of $u$. By $T_x$ let us denote the subtree induced by a vertex $x$ and its descendants in the rooted tree $T$.

First assume that $d_T(v) \geq 5$. Let $T' = T - T_v$. Let us observe that there exists a $\gamma_2(T)$-set that does not contain the vertex $v$. Let $D$ be such a set. Observation 2 implies that all leaves adjacent to $v$ belong to the set $D$. Observe that after removing them from the set $D$ we get a 2DS of the tree $T'$. Therefore $\gamma_2(T') \leq \gamma_2(T)$
The assumption \( b_2(T) \in \{1, 2\} \) implies that there exists \( E' \subseteq E(T) \) such that \(|E'| = b_2(T) \leq 2\) and \( \gamma_2(T - E') > \gamma_2(T) \). In every \( \gamma_2(T)\)-set the vertex \( v \) has at least four neighbors. This implies that the set \( E' \) does not contain any edge incident to \( v \). By \( T''(T''') \), respectively, we denote the component of \( T - E' \) (\( T'' - E' \), respectively) which contains the vertex \( u \). Let \( D'' \) be any \( \gamma_2(T'')\)-set. It is easy to observe that the elements of the set \( D'' \) together with the leaves adjacent to \( v \) form a 2DS of the tree \( T'' \). Thus \( \gamma_2(T'') \leq \gamma_2(T''') + d_T(v) - 1 \). Now we get \( \gamma_2(T' - E') = \gamma_2(T' - E' - T''') + \gamma_2(T''') \geq \gamma_2(T' - E' - T''') + \gamma_2(T'' - d_T(v) + 1 = \gamma_2(T' - E') + \gamma_2(T'') \geq \gamma_2(T' - E') + d_T(v) + 1 > \gamma_2(T) - d_T(v) + 1 \geq \gamma_2(T') \). This implies that \( 0 \neq b_2(T') \leq |E'| \leq 2 \), and consequently, \( b_2(T') \in \{1, 2\} \).

By the inductive hypothesis we have \( T' \in T \). The tree \( T \) can be obtained from \( T' \) by Operation \( O_1 \). Thus \( T \in T \).

Now assume that \( d_T(v) = 4 \). The leaves adjacent to \( v \) and different from \( t \) are denoted by \( a \) and \( b \). If no \( \gamma_2(T)\)-set contains the vertex \( u \), then \( T \in T_0 \) as \( u \) is adjacent to a star \( K_{1,3} \) through the central vertex. Now assume that there exists a \( \gamma_2(T)\)-set that contains the vertex \( u \). Let \( D \) be such a set. By Observation 2 we have \( t, a, b \in D \). The set \( D \) is minimal, and thus \( v \notin D \). Let \( T' = T - T_v \). Observe that \( D \setminus \{t, a, b\} \) is a 2DS of the tree \( T' \). Therefore \( \gamma_2(T') \leq \gamma_2(T) - 3 \). The assumption \( b_2(T) \in \{1, 2\} \) implies that there exists \( E' \subseteq E(T) \) such that \(|E'| = b_2(T) \leq 2 \) and \( \gamma_2(T - E') > \gamma_2(T) \). The vertex \( v \) has four neighbors in \( D \), and thus the set \( E' \) does not contain any edge incident to \( v \). By \( T'''(T'''), \) respectively, we denote the component of \( T - E' \) (\( T''' - E' \), respectively) which contains the vertex \( u \). Let \( D''' \) be any \( \gamma_2(T'''')\)-set. It is easy to observe that \( D''' \cup \{t, a, b\} \) is a 2DS of the tree \( T''' \). Thus \( \gamma_2(T''') \leq \gamma_2(T''''') + 3 \). Now we get \( \gamma_2(T' - E') = \gamma_2(T' - E' - T''''') + \gamma_2(T''''') \geq \gamma_2(T' - E' - T''''') + \gamma_2(T''') - 3 = \gamma_2(T' - E' - T''') + \gamma_2(T''') - 3 = \gamma_2(T - E') - 3 > \gamma_2(T) - 3 \geq \gamma_2(T) \). Now we conclude that \( b_2(T') \in \{1, 2\} \).

By the inductive hypothesis we have \( T' \in T \). The tree \( T \) can be obtained from \( T' \) by Operation \( O_1 \). Thus \( T \in T \).

Now assume that \( d_T(v) = 3 \). The vertex \( v \) is a strong support vertex of degree three. Thus \( T \in T_0 \subseteq T \).

Now assume that \( d_T(v) = 2 \). First assume that some child of \( u \) other than \( v \), say \( x \), is a support vertex. It suffices to consider only the possibility when \( x \) is adjacent to exactly one leaf. The vertex \( u \) is adjacent to at least two support vertices of degree two. Thus \( T \in T_0 \subseteq T \).

Now assume that every child of \( u \) different from \( v \) is a leaf. Let \( T' \) be a tree that differs from \( T - T_u \) only in that it has the vertex \( u \). Let us observe that there exists a \( \gamma_2(T)\)-set that contains the vertex \( u \). Let \( D \) be such a set. Observation 2 implies that all leaves of \( T_u \) belong to the set \( D \). Since \( D \) is minimal, it does not contain any vertex, which neither exists in the tree \( T' \) nor is a leaf. It is easy to observe that after removing from \( D \) all leaves of \( T_u \) we get a 2DS of the tree \( T' \). Therefore \( \gamma_2(T') \leq \gamma_2(T) - d_T(u) + 1 \). The assumption \( b_2(T) \in \{1, 2\} \) implies that there exists \( E' \subseteq E(T) \) such that \(|E'| = b_2(T) \leq 2 \) and \( \gamma_2(T - E') > \gamma_2(T) \). Let us observe that the set \( E' \) does not contain any edge incident to a leaf adjacent to \( u \). Assume that \( E' \) contains \( uw \) or \( vt \). This implies that no \( \gamma_2(T)\)-set contains the vertex \( v \). Let us observe that \( \gamma_2(T' - uw) > \gamma_2(T') \). This implies that \( b_2(T') = 1 \). Now assume that the set \( E' \) does not contain any edge of \( T_u \). By \( T''(T''') \), respectively, we denote the component of \( T - E' \) (\( T'' - E' \), respectively) which contains the vertex \( u \). Let \( D'' \) be any \( \gamma_2(T'')\)-set. By Observation 2 we have \( u \in D'' \). It is easy to observe that the elements of the set \( D'' \) together with all leaves of \( T_u \) form a 2DS of the tree \( T_u \). Thus \( \gamma_2(T_u) \leq \gamma_2(T''') + d_T(u) - 1 \). Now we get \( \gamma_2(T' - E') = \gamma_2(T' - E' - T''') + \gamma_2(T''') \geq \gamma_2(T' - E' - T''') + \gamma_2(T''') + 1 = \gamma_2(T' - E') + d_T(u) + 1 \geq \gamma_2(T) + d_T(u) + 1 \geq \gamma_2(T') \). Now we conclude that
REFERENCES

\[ b_2(T') \in \{1, 2\} \]. By the inductive hypothesis we have \( T' \in T \). The tree \( T \) can be obtained from \( T'' \) by Operation \( O_2 \). Thus \( T \in T \).

As an immediate consequence of Lemmas 15 and 16, we have the following characterization of trees with 2-bondage number equaling one or two.

Theorem 2.1 Let \( T \) be a tree. Then \( b_2(T) \in \{1, 2\} \) if and only if \( T \in T \).

References

[5] K. Hartnell and D. Rall, A characterization of trees in which no edge is essential to the domination number, Ars Combinatoria 33 (1992), 65–76.