On trees with equal 2-domination and 2-outer-independent domination numbers

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Abstract

For a graph $G = (V, E)$, a subset $D \subseteq V(G)$ is a 2-dominating set if every vertex of $V(G) \setminus D$ has at least two neighbors in D, while it is a 2-outer-independent dominating set if additionally the set $V(G) \setminus D$ is independent. The 2-domination (2-outer-independent domination, respectively) number of G, is the minimum cardinality of a 2-dominating (2-outer-independent dominating, respectively) set of G. We characterize all trees with equal 2-domination and 2-outer-independent domination numbers.

Keywords: 2-domination, 2-outer-independent domination, tree.

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1 Introduction

Let $G = (V, E)$ be a graph. By the neighborhood of a vertex v of G we mean the set $N_G(v) = \{u \in V(G) \colon uv \in E(G)\}$. The degree of a vertex v, denoted by $d_G(v)$, is the cardinality of its neighborhood. By a leaf we mean a vertex of degree one, while a support vertex is a vertex adjacent to a leaf. We say that a subset of $V(G)$ is independent if there is no edge between any two vertices of this set. A path on n vertices we denote by P_n. By a star we mean a connected graph in which exactly one vertex has degree greater than one. Let uv be an edge of a graph G. By subdividing the edge uv we mean removing it, and adding a new vertex, say x, along with two new edges ux and xv.

A subset $D \subseteq V(G)$ is a dominating set of G if every vertex of $V(G) \setminus D$ has a neighbor in D, while it is a 2-dominating set, abbreviated 2DS, of G if

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every vertex of $V(G) \setminus D$ has at least two neighbors in D. The domination (2-domination, respectively) number of G, denoted by $\gamma(G)$ ($\gamma_2(G)$, respectively), is the minimum cardinality of a dominating (2-dominating, respectively) set of G. A 2-dominating set of G of minimum cardinality is called a $\gamma_2(G)$-set. Note that 2-domination is a type of multiple domination in which each vertex, which is not in the dominating set, is dominated at least k times for a fixed positive integer k. Multiple domination in graphs was introduced by Fink and Jacobson [2], and was further studied for example in [1, 3, 4]. For a comprehensive survey of domination in graphs, see [5].

A subset $D \subseteq V(G)$ is a 2-outer-independent dominating set, abbreviated 2OIDS, of G if every vertex of $V(G) \setminus D$ has at least two neighbors in D, and the set $V(G) \setminus D$ is independent. The 2-outer-independent domination number of G, denoted by $\gamma_2^{oi}(G)$, is the minimum cardinality of a 2-outer-independent dominating set of G. A 2-outer-independent dominating set of G of minimum cardinality is called a $\gamma_2^{oi}(G)$-set. The study of 2-outer-independent domination in graphs was initiated in [6].

We characterize all trees with equal 2-domination and 2-outer-independent domination numbers.

2 Results

We begin with the following three straightforward observations.

Observation 1 *For every graph G we have $\gamma_2^{oi}(G) \geq \gamma_2(G)$.*

Observation 2 *Every leaf of a graph G is in every $\gamma_2(G)$-set and in every $\gamma_2^{oi}(G)$-set.*

Observation 3 *For every path there is a minimum 2-dominating set that contains all vertices that are at even distance from one of the leaves.*

Let T be a tree. We say that two vertices of T of degree at least three are linked, if all interior vertices of the path joining them in T have degree two. Then the path is called a link. Paths joining leaves of T to the closest vertices of degree at least three we call chains. The length of a link or a chain is the number of its edges. A link or a chain is even (odd, respectively) if its length is even (odd, respectively). We say that a vertex of T of degree at least three, say x, is within even range of a leaf, if there is a leaf, say y, such that all links and chains of the path joining x and y in T are even.

Let $\mathcal{T}_0$ be a family of trees in which for every pair of adjacent vertices of degree at least three, at least one of them is within even range of a leaf.

Lemma 4 *If $T \in \mathcal{T}_0$, then $\gamma_2^{oi}(T) = \gamma_2(T)$.*

Proof. Observation 3 implies that for every tree there is a minimum 2-dominating set that contains all vertices of degree at least three that are within even range of a leaf. Let D be such a set for the tree T. Suppose that some two adjacent vertices of T, say x and y, do not belong to the set D. Since $T \in \mathcal{T}_0$, at least one of them has degree two. This is a contradiction as that vertex must have at least two neighbors in D. We now conclude that for every pair of adjacent vertices of T, the set D contains at least one of them. Thus $V(T) \setminus D$ is an independent set. Consequently, D is a 2OIDS of the tree T. Therefore $\gamma_2^{oi}(T) \leq \gamma_2(T)$. On the other hand, by Observation 1 we have $\gamma_2^{oi}(T) \geq \gamma_2(T)$. ■

We characterize all trees with equal 2-domination and 2-outer-independent domination numbers. For this purpose we introduce a family $\mathcal{T}$ of trees $T = T_k$ that can be obtained as follows. Let $T_1 \in \mathcal{T}_0$. If k is a positive integer, then T_{k+1} can be obtained recursively from T_k by the following operation. Let x be a vertex of T_k, which belongs to some $\gamma_2^{oi}(T)$-set. Let y be the central vertex of a star, each edge of which can be subdivided any non-negative even number of times. Then join the vertices x and y.

For checking whether a given vertex of a tree belongs to some of its minimum 2-outer-independent dominating sets, let us consider the following algorithm, which labels vertices of a tree T as taken, omitted and undecided. Initialize by calling every leaf taken and every other vertex undecided. Root T at a non-leaf vertex, say r. Let $u \neq r$ be a vertex of T, which has not already been decided, and such that all its children have been decided. If some child of u has been omitted, then take u. Otherwise omit u and take its parent.

Proposition 5 *Let T be a tree, and let v be a vertex of T. There exists a $\gamma_2^{oi}(T)$-set containing the vertex v if and only if v is a leaf or, rooting T at v, the above algorithm labels at least one child of v as omitted.*

We now prove that for every tree of the family $\mathcal{T}$, the 2-domination and the 2-outer-independent domination numbers are equal.

Lemma 6 *If $T \in \mathcal{T}$, then $\gamma_2^{oi}(T) = \gamma_2(T)$.*

Proof. We use the induction on the number k of operations performed to construct the tree T. If $T \in \mathcal{T}_0$, then by Lemma 4 we have $\gamma_2^{oi}(T) = \gamma_2(T)$. Let k be a positive integer. Assume that the result is true for every $T' = T_k$ of the family $\mathcal{T}$ constructed by $k-1$ operations. Let x be a vertex of T' to which is attached the new tree T_1. It is easy to notice that $\gamma_2^{oi}(T_1) = \gamma_2(T_1)$. The vertices of T_1 at odd distance from the vertex of maximum degree, say y, form a $\gamma_2^{oi}(T_1)$-set. Let D' be a $\gamma_2^{oi}(T')$-set that contains the vertex x. It is easy to observe that the elements of the set D' together with the vertices of T_1 at odd distance from y, form a 2OIDS of the tree T. Thus $\gamma_2^{oi}(T) \leq \gamma_2^{oi}(T') + \gamma_2^{oi}(T_1)$. Now let us observe that there exists a $\gamma_2(T)$-set that does not contain the vertex y and the vertices of T_1 at even distance from y. Let D be such a set. Notice that all vertices of T_1 at odd distance from y belong to the set D. Observe that $D \cap V(T')$ is a 2DS of the tree T'. Therefore $\gamma_2(T') \leq \gamma_2(T) - \gamma_2(T_1)$. We now get $\gamma_2^{oi}(T) \leq \gamma_2^{oi}(T') + \gamma_2^{oi}(T_1) = \gamma_2(T') + \gamma_2(T_1) \leq \gamma_2(T)$. This implies that $\gamma_2^{oi}(T) = \gamma_2(T)$. ∎

We now prove that if the 2-domination and the 2-outer-independent domination numbers of a tree are equal, then the tree belongs to the family $\mathcal{T}$.

Lemma 7 *Let T be a tree. If $\gamma_2^{oi}(T) = \gamma_2(T)$, then $T \in \mathcal{T}$.*

Proof. The result we obtain by the induction on the order n of the tree T. Assume that the lemma is true for every tree T' of order $n' < n$. If at most one vertex of T has degree at least three, then it follows from the definition of the family $\mathcal{T}_0$ that $T \in \mathcal{T}_0 \subseteq \mathcal{T}$ as in the tree T there is no pair of adjacent vertices of degree at least three. Now assume that at least two vertices of T have degree at least three. Let x be a vertex of T of degree at least three, which is adjacent to exactly one link. Thus x is adjacent to at least two chains. First assume that some of them is even. Let T_x be the tree induced by the vertex x and the chains adjacent to x. Let S be the set of vertices of $V(T_x) \setminus \{x\}$ that are leaves or are at even distance from x. Let T' be a tree obtained from T by replacing T_x with a path P_3, say xyz, where z is the leaf. Let D' be a $\gamma_2(T')$-set that contains the vertices x and z. It is easy to observe that $D' \cup S \setminus \{z\}$ is a 2DS of the tree T. Thus $\gamma_2(T) \leq \gamma_2(T') + |S| - 1$. Now let us observe that there exists a $\gamma_2^{oi}(T)$-set that does not contain the vertices of T_x, which are not leaves and are at odd distance from x. Let D be such a set. Observe that $\{z\} \cup D \cap V(T')$ is a 2OIDS of the tree T'. Therefore $\gamma_2^{oi}(T') \leq \gamma_2^{oi}(T) - |S| + 1$. We now get $\gamma_2^{oi}(T') \leq \gamma_2^{oi}(T) - |S| + 1 = \gamma_2(T) - |S| + 1 \leq \gamma_2(T')$. This implies that $\gamma_2^{oi}(T') = \gamma_2(T')$. By the inductive hypothesis we have $T' \in \mathcal{T}$. It follows from the definition of the family $\mathcal{T}$ that $T \in \mathcal{T}$.

Now assume that all chains adjacent to x are odd. Let T_x be the tree induced by the vertex x and the chains adjacent to x. The neighbor of x that does not belong to $V(T_x)$ we denote by k. Let S be the set of vertices of T_x that are at odd distance from x. Let $T' = T - T_x$. Let D' be any $\gamma_2(T')$-set. It is easy to observe that $D' \cup S$ is a 2DS of the tree T. Thus $\gamma_2(T) \leq \gamma_2(T') + |S|$. Now let us observe that there exists a $\gamma_2^{oi}(T)$-set that does not contain the vertex x and the vertices of T_x at even distance from x. Let D be such a set. The set $V(T) \setminus D$ is independent, thus $k \in D$. Observe that $D \setminus S$ is a 2OIDS of the tree T' of cardinality $\gamma_2^{oi}(T) - |S|$, and which contains the vertex k. Therefore $\gamma_2^{oi}(T') \leq \gamma_2^{oi}(T) - |S|$. We now get $\gamma_2^{oi}(T') \leq \gamma_2^{oi}(T) - |S| = \gamma_2(T) - |S| \leq \gamma_2(T')$. This implies that $\gamma_2^{oi}(T') = \gamma_2(T')$. By the inductive hypothesis we have $T' \in \mathcal{T}$. Moreover, there exists a $\gamma_2^{oi}(T')$-set that contains the vertex k. The tree T_x is obtained from a star by subdividing each one of its edges a non-negative even number of times. The tree T can be obtained from T' by attaching the tree T_x by joining the central vertex to the vertex k. Thus $T \in \mathcal{T}$. ■

As an immediate consequence of Lemmas 6 and 7, we have the following characterization of trees with equal 2-domination and 2-outer-independent domination numbers.

Theorem 8 *Let T be a tree. Then $\gamma_2^{oi}(T) = \gamma_2(T)$ if and only if $T \in \mathcal{T}$.*

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