Trees having many minimal dominating sets

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Abstract

We disprove a conjecture by Skupień that every tree of order \( n \) has at most \( 2^n/2 \) minimal dominating sets. We construct a family of trees of both parities of the order for which the number of minimal dominating sets exceeds \( 1.4167^n \). We also provide an algorithm for listing all minimal dominating sets of a tree in time \( O(1.4656^n) \). This implies that every tree has at most \( 1.4656^n \) minimal dominating sets.

Keywords: minimal dominating set, tree, combinatorial bound, exponential algorithm, listing algorithm.

1 Introduction

Let \( G = (V, E) \) be a graph. The order of a graph is the number of its vertices. By the neighborhood of a vertex \( v \) of \( G \) we mean the set \( N_G(v) = \{ u \in V(G) : uv \in E(G) \} \). The degree of a vertex \( v \), denoted by \( d_G(v) \), is the cardinality of its neighborhood. By a leaf we mean a vertex of degree one, while a support vertex is a vertex adjacent to a leaf. We say that a support vertex is strong (weak, respectively) if it is adjacent to at least two leaves (exactly one leaf, respectively). The distance between two vertices of a graph is the number of edges in a shortest path connecting them. The eccentricity of a vertex is the greatest distance between it and any other vertex. The diameter of a graph \( G \), denoted by \( \text{diam}(G) \), is the maximum eccentricity among all vertices of \( G \). Denote by \( P_n \) a path on \( n \) vertices. By a star we mean a connected graph in which exactly one vertex has degree greater than one.

A subset \( D \subseteq V(G) \) is a dominating set of \( G \) if every vertex of \( V(G) \setminus D \) has a neighbor in \( D \). A dominating set \( D \) is minimal if no proper subset of \( D \) is a dominating set of \( G \). For a comprehensive survey of domination in graphs, see [9, 10].

One of the typical questions in graph theory is how many subgraphs of a given property can a graph on \( n \) vertices have. For example, the famous Moon and Moser theorem [14] says that every graph on \( n \) vertices has at most \( 3^{n/3} \) maximal independent sets.

Combinatorial bounds are of interest not only on their own, but also because they are used for algorithm design as well. Lawler [13] used the Moon-Moser bound on the number of maximal independent sets to construct an \( (1 + \sqrt{3})^n \cdot n^{O(1)} \) time graph coloring algorithm, which was the fastest one known for twenty-five years. In 2003 Eppstein [5] reduced the running time of a graph coloring to \( O(2.4151^n) \). In 2006 the running time was reduced [1, 12] to \( O(2^n) \). For an overview of the field, see [7].

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Fomin et al. [6] constructed an algorithm for listing all minimal dominating sets of a graph on \( n \) vertices in time \( O(1.7159^n) \). There were also given graphs \((n/6)\) disjoint copies of the octahedron) having \( 15^{n/6} \approx 1.5704^n \) minimal dominating sets. This establishes a lower bound on the running time of an algorithm for listing all minimal dominating sets of a given graph.

The number of maximal independent sets in a tree was investigated in [16]. Couturier et al. [4] considered minimal dominating sets in various classes of graphs. The authors of [11] investigated the enumeration of minimal dominating sets in graphs.

Bród and Skupień [2] gave bounds on the number of dominating sets of a tree. They also characterized the extremal trees. The authors of [3] investigated the number of minimal dominating sets in trees containing all leaves.

Skupień [15] conjectured that every tree of order \( n \) has at most \( 2^{n/2} \) minimal dominating sets. It turns out that there are trees having more than \( 2^{n/2} \) minimal dominating sets, which contradicts the conjecture. We construct a family of trees of both parities of the order for which the number of minimal dominating sets exceeds \( 1.4167^n \), thus exceeding \( 2^{n/2} \). Since \( 2^{n/2} \) is not a correct upper bound on the number of minimal dominating sets of a tree, we aim to prove a correct one. We provide an algorithm for listing all minimal dominating sets of a tree of order \( n \) in time \( O(1.4656^n) \). This implies that every tree has at most \( 1.4656^n \) minimal dominating sets.

## 2 Disproof of the conjecture

Now we give an infinite family of trees \( \{ T_k \}_{k=1}^\infty \) of odd and even order for which the number of minimal dominating sets exceeds \( 1.4167^n > 2^{n/2} \). Let \( T_1 \) be the tree given in Figure 1. Let \( T_{k+1} \) be a tree obtained from \( T_k \) by adding an edge connecting one of its support vertices to a support vertex of \( T_1 \).

![Figure 1: The tree \( T_1 \) with 27 vertices](image)

Now we calculate the number of minimal dominating sets of a tree \( T_k \). We root the tree \( T_1 \) at the vertex \( y \). Let \( D \) be a minimal dominating set of \( T_1 \). Observe that for every leaf, either it belongs to the set \( D \) or its neighbor belongs to \( D \). First assume that some child of \( x \) belongs to the set \( D \), and also some child of \( z \) belongs to \( D \). Thus both vertices \( x \) and \( z \) are already dominated. Since the vertex \( y \) has to be dominated, we have either \( x \in D \) or \( y \in D \) or \( z \in D \). There are \( 3 \cdot (2^6 - 1)^2 \) such minimal dominating sets. Now assume that some child of \( x \) belongs to the set \( D \), while no child of \( z \) belongs to \( D \). We have either \( y \in D \) or \( z \in D \) as the vertices \( y \) and \( z \) have to be dominated. There are \( 2 \cdot (2^6 - 1) \) such minimal dominating sets. The possibility when some child of \( z \) belongs to the set \( D \) and no child of \( x \) belongs to \( D \) is similar. If no child of \( x \) and \( z \) belongs to the set \( D \), then either \( x, z \in D \) or \( y \in D \). Now we conclude that the tree \( T_1 \) has \( 3 \cdot (2^6 - 1)^2 + 2 \cdot 2 \cdot (2^6 - 1) + 2 = 12161 \) minimal dominating sets. While constructing trees of the family \( \{ T_k \}_{k=1}^\infty \), support vertices are connected by edges. Let us observe that every support
vertex is already dominated, as it is adjacent to a leaf. This implies that the new edges do not influence the number of minimal dominating sets. Consequently, the tree $T_k$ has $12161^k$ minimal dominating sets. We have $\sqrt[3]{12161^k} = \sqrt[3]{12161} \approx 4.41667 > 4.1467$. This implies that $\Omega(4.1467^n)$ is a lower bound on the running time of an algorithm for listing all minimal dominating sets of a given tree of order $n$.

Independently, in 2011 Górska [8] found an infinite family of $T_i$-like trees, in which the vertices $x$ and $z$ can have arbitrarily equitably many children. This left the conjecture open only for trees of even order.

3 Listing algorithm

In this section we describe an algorithm, which lists all minimal dominating sets of a given input tree $T$. Denote by $\mathcal{F}(T)$ the family of sets returned by the algorithm.

Algorithm

Let $T$ be a tree. If diam($T$) = 0, then $T = P_1 = v_1$. Let $\mathcal{F}(T) = \{\{v_1\}\}$. If diam($T$) = 1, then $T = P_2 = v_1v_2$. Let $\mathcal{F}(T) = \{\{v_1\}, \{v_2\}\}$. If diam($T$) = 2, then $T$ is a star. Denote by $x$ the support vertex of $T$. Let $\mathcal{F}(T) = \{\{x\}, V(T) \setminus \{x\}\}$.

Now assume that diam($T$) ≥ 3. First assume that some support vertex of $T$, say $x$, is strong. Let $y$ be a leaf adjacent to $x$. Let $T' = T - y$, and let

$$\mathcal{F}(T) = \{D' : x \in D' \in \mathcal{F}(T')\} \cup \{D' \cup \{y\} : x \notin D' \in \mathcal{F}(T')\}.$$ 

Now assume that every support vertex of $T$ is weak. We root $T$ at a vertex $r$ of maximum eccentricity diam($T$). Let $t$ be a leaf at maximum distance from $r$, $v$ be the parent of $t$, and $u$ be the parent of $v$ in the rooted tree. If diam($T$) ≥ 4, then let $w$ be the parent of $u$. Denote by $T_x$ the subtree induced by a vertex $x$ and its descendants in the rooted tree $T$.

Assume that some child of $u$, say $x$, is a leaf. Let $T' = T - T_v$, and let

$$\mathcal{F}(T) = \{D' \cup \{v\}, D' \cup \{t\} : D' \in \mathcal{F}(T')\}.$$ 

Now assume that every child of $u$ is a support vertex. The children of $u$ are denoted by $k_1, k_2, \ldots, k_{d_T(u)}$, where $k_1 = v$. Let $l_i$ mean the leaf adjacent to $k_i$. Let $p_i \in \{k_i, l_i\}$. Denote by $w$ the parent of $u$. The neighbors of $w$ other than $u$ we denote by $m_1, m_2, \ldots, m_{d_T(w)}$. Let $T' = T - T_{k_1} - T_{k_2} - \ldots - T_{k_{d_T(u)}}$ and $T'' = T - T_u$. The components of $T'' - w$ are denoted by $T_1, T_2, \ldots, T_{d_T(w)}$, where $m_i \in V(T_i)$.

Let $\mathcal{F}(T)$ be a family as follows,

$$\begin{align*}
\mathcal{F}(T) &= \left\{ D' \cup \bigcup_{1 \leq i \leq d_T(u) - 1} \{l_i\} : D' \in \mathcal{F}(T') \right\} \\
&\cup \left\{ D'' \cup \bigcup_{1 \leq i \leq d_T(u) - 1} \{p_i\} : D'' \in \mathcal{F}(T'') \text{ and } \exists_j p_j = k_j \right\} \\
&\cup \left\{ \bigcup_{1 \leq i \leq d_T(w) - 1} D_i \cup \{u\} \cup \bigcup_{1 \leq j \leq d_T(w) - 1} \{p_j\} : m_j \notin D_j \in \mathcal{F}(T_j) \text{ and } \exists \ell p_\ell = k_\ell \right\},
\end{align*}$$

where the third component is ignored if $w$ is adjacent to a leaf.
4 Bounding the number of minimal dominating sets

Now we prove that the running time of the algorithm from the previous section is $O(1.4656^n)$.

**Theorem 1** For every tree $T$ of order $n$, the algorithm from the previous section lists all minimal dominating sets in time $O(1.4656^n)$.

**Proof.** We prove that the running time of the algorithm is $O(1.4656^n)$. Moreover, we prove that the number of minimal dominating sets is at most $\alpha^n$, where $\alpha \approx 1.46557 < 1.4656$ is the positive solution of the equation $x^4 - x^2 - x - 1 = 0$.

We proceed by induction on the number $n$ of vertices of a tree $T$. If $\text{diam}(T) = 0$, then $T = P_1 = v_1$. Obviously, $\{v_1\}$ is the only minimal dominating set of the path $P_1$. We have $n = 1$ and $|F(T)| = 1$. Obviously, $1 < \alpha$. If $\text{diam}(T) = 1$, then $T = P_2 = v_1v_2$. It is easy to see that $\{v_1\}$ and $\{v_2\}$ are the only two minimal dominating sets of the path $P_2$. We have $n = 2$ and $|F(T)| = 2$. We also have $2 < \alpha^2$ as $\alpha > \sqrt{2}$. If $\text{diam}(T) = 2$, then $T$ is a star. Denote by $x$ the support vertex of $T$. It is easy to observe that $\{x\}$ and $V(T) \setminus \{x\}$ are the only two minimal dominating sets of the tree $T$. We have $n \geq 3$ and $|F(T)| = 2$. Consequently, $2 < \alpha^2 < \alpha^n$.

Now assume that $\text{diam}(T) \geq 3$. First assume that some support vertex of $T$, say $x$, is strong. Let $y$ be a leaf adjacent to $x$. Let $T' = T - y$. Let $D'$ be a minimal dominating set of the tree $T'$. If $x \in D'$, then it is easy to see that $D'$ is a minimal dominating set of the tree $T$. Now assume that $x \notin D'$. It is easy to observe that $D' \cup \{y\}$ is a minimal dominating set of the tree $T$. Thus all elements of the family $F(T)$ are minimal dominating sets of the tree $T$. Now let $D$ be any minimal dominating set of the tree $T$. Clearly, either the vertex $x$ belongs to the set $D$ or all leaves adjacent to $x$ belong to the set $D$. If $x \in D$, then it is easy to see that $D$ is a minimal dominating set of the tree $T'$. By the inductive hypothesis we have $D \in F(T')$. Now assume that $x \notin D$. It is evident that $D \setminus \{y\}$ is a minimal dominating set of the tree $T'$. By the inductive hypothesis we have $D \setminus \{y\} \in F(T')$. Therefore the family $F(T)$ contains all minimal dominating sets of the tree $T$. Now we get $|F(T)| = |\{D' : x \in D' \in F(T')\}| + |\{D' \cup \{y\} : x \notin D' \in F(T')\}| = |F(T')| \leq \alpha^{n-1} < \alpha^n$.

Henceforth, we can assume that every vertex of $T$ is weak.

We now root $T$ at a vertex $r$ of maximum eccentricity $\text{diam}(T)$. Let $t$ be a leaf at maximum distance from $r$, $v$ be the parent of $t$, and $u$ be the parent of $v$ in the rooted tree. If $\text{diam}(T) \geq 4$, then let $w$ be the parent of $u$. Denote by $T_u$ the subtree induced by a vertex $x$ and its descendants in the rooted tree $T$.

Assume that some child of $u$, say $x$, is a leaf. Let $T' = T - T_v$. Let us observe that all elements of the family $F(T)$ are minimal dominating sets of the tree $T$. Now let $D$ be any minimal dominating set of the tree $T$. We have either $v \in D$ or $t \in D$ as the vertex $t$ has to be dominated and the set $D$ is minimal. Similarly, either $u \in D$ or $x \in D$. If $t \in D$, then observe that $D \setminus \{t\}$ is a minimal dominating set of the tree $T'$. By the inductive hypothesis we have $D \setminus \{t\} \in F(T')$. Now assume that $v \in D$. Let us observe that $D \setminus \{v\}$ is a minimal dominating set of the tree $T''$ as the vertex $u$ is still dominated. By the inductive hypothesis we have $D \setminus \{v\} \in F(T'')$. Therefore the family $F(T)$ contains all minimal dominating sets of the tree $T$. Now we get $|F(T)| = 2|F(T')| \leq 2 \cdot \alpha^{n-2} < \alpha^2 \cdot \alpha^{n-2} = \alpha^n$.

Now assume that every child of $u$ is a support vertex. We use the same notation as in the description of the algorithm. Let $T'' = T - T_{k_1} - T_{k_2} - \ldots - T_{k_{d(u)-1}}$ and $T'' = T - T_u$. The components of $T'' - w$ we denote by $T_1, T_2, \ldots, T_{d(u)-1}$, where $m_i \in V(T_i)$. It is not very difficult to verify that all elements of the family $F(T)$ are minimal dominating sets of the tree $T$. Now let $D$ be any minimal dominating set of the tree $T$. If all leaves of $T_u$ belong to the set $D$, then observe that $D \setminus \{l_1, l_2, \ldots, l_{d(u)-1}\}$ is a minimal dominating set of the tree $T''$. Now assume that some support vertex of $T_u$ belongs to the set $D$. If $u \notin D$, then observe that $D \cap V(T'')$ is a minimal dominating set of the tree $T''$. Now assume that $u \in D$. Let us observe that neither $w$ nor any of its neighbors other than $u$ belongs to the set $D$, otherwise $D \setminus \{u\}$ is
a dominating set of the tree $T$, a contradiction to the minimality of $D$. Let us observe that $D \cap V(T_i)$ is a minimal dominating set of the tree $T_i$, which does not contain the vertex $m_i$. Now we get

$$|\mathcal{F}(T)| \leq |\mathcal{F}(T')| + (2^{d_{T}(u)}-1 - 1) \left( |\mathcal{F}(T'')| + \prod_{1 \leq i \leq d_{T}(u)-1} |D_i \in \mathcal{F}(T_i) : m_i \notin D_i| \right)$$

$$< |\mathcal{F}(T')| + (2^{d_{T}(u)}-1 - 1) \left( |\mathcal{F}(T'')| + \prod_{1 \leq i \leq d_{T}(u)-1} |\mathcal{F}(T_i)| \right)$$

$$\leq \alpha^{n-2d_{T}(u)+2} + (2^{d_{T}(u)}-1) (\alpha + 1 \leq \alpha^{n-2d_{T}(u)+1} + \alpha^{n-2d_{T}(u)}) \leq \alpha^n.$$

To show that $\alpha^{n-2d_{T}(u)+2} + (2^{d_{T}(u)}-1) (\alpha + 1 \leq \alpha^{n-2d_{T}(u)+1} + \alpha^{n-2d_{T}(u)}) \leq \alpha^n$, it suffices to show that $\alpha^2 + (2^{d_{T}(u)}-1) (\alpha + 1 \leq \alpha^{2d_{T}(u)}.$ We prove this by induction on the degree of the vertex $u$. For $d_{T}(u) = 2$ we have $\alpha^2 + (2^{d_{T}(u)}-1) (\alpha + 1 = \alpha^2 + \alpha + 1 = \alpha^4 = \alpha^{2d_{T}(u)}.$ Now we prove that if the inequality $\alpha^2 + (2^{d_{T}(u)}-1) (\alpha + 1 \leq \alpha^{2d_{T}(u)}$ is satisfied for an integer $k = d_{T}(u) \geq 2$, then it is also satisfied for $k + 1$. We have $\alpha^2 + (2^k - 1) (\alpha + 1 = \alpha^4 - \alpha - 1 + (2^k - 1) (\alpha + 1 = \alpha^4 + (2^k - 2) (\alpha + 1 = \alpha^4 + 2^{2k-1} (\alpha + 1 = \alpha^4 - 2\alpha^2 + 2(\alpha^2 - 2) (\alpha + 1) \leq \alpha^4 - 2\alpha^2 + 2\alpha^2 - 2\alpha - 2 < \alpha^{2k+2} + \alpha^2 (\alpha - 2) + \alpha^2 = \alpha^{2k+2} + (\alpha^2 - 2) (\alpha^2 - 2) < \alpha^{2k+2}.$

It follows from the proof of the previous theorem that any tree of order $n$ has less than $1.4656^n$ minimal dominating sets.

**Corollary 2** Every tree of order $n$ has at most $\alpha^n$ minimal dominating sets, where $\alpha \approx 1.46557$ is the positive solution of the equation $x^4 - x^2 - x - 1 = 0$.

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**References**


