Trees having many minimal dominating sets

Marcin Krzywkowski*
e-mail: marcin.krzywkowski@gmail.com

Faculty of Electronics, Telecommunications and Informatics
Gdańsk University of Technology
Narutowicza 11/12, 80–233 Gdańsk, Poland

Abstract

We disprove a conjecture by Skupień that every tree of order n has at most $2^{n/2}$ minimal dominating sets. We construct a family of trees of both parities of the order for which the number of minimal dominating sets exceeds 1.4167^n. We also provide an algorithm for listing all minimal dominating sets of a tree in time $\mathcal{O}(1.4656^n)$. This implies that every tree has at most 1.4656^n minimal dominating sets.
Keywords: minimal dominating set, tree, combinatorial bound, exponential algorithm, listing algorithm.

1 Introduction

Let $G = (V, E)$ be a graph. The *order* of a graph is the number of its vertices. By the *neighborhood* of a vertex v of G we mean the set $N_G(v) = \{u \in V(G)\colon uv \in E(G)\}$. The *degree* of a vertex v, denoted by $d_G(v)$, is the cardinality of its neighborhood. By a *leaf* we mean a vertex of degree one, while a *support vertex* is a vertex adjacent to a leaf. We say that a support vertex is *strong* (*weak*, respectively) if it is adjacent to at least two leaves (exactly one leaf, respectively). The *distance* between two vertices of a graph is the number of edges in a shortest path connecting them. The *eccentricity* of a vertex is the greatest distance between it and any other vertex. The *diameter* of a graph G, denoted by $\mathrm{diam}(G)$, is the maximum eccentricity among all vertices of G. Denote by P_n a *path* on n vertices. By a *star* we mean a connected graph in which exactly one vertex has degree greater than one.

A subset $D \subseteq V(G)$ is a *dominating set* of G if every vertex of $V(G) \setminus D$ has a neighbor in D. A dominating set D is *minimal* if no proper subset of D is a dominating set of G. For a comprehensive survey of domination in graphs, see [9, 10].

One of the typical questions in graph theory is how many subgraphs of a given property can a graph on n vertices have. For example, the famous Moon and Moser theorem [14] says that every graph on n vertices has at most $3^{n/3}$ maximal independent sets.

Combinatorial bounds are of interest not only on their own, but also because they are used for algorithm design as well. Lawler [13] used the Moon-Moser bound on the number of maximal independent sets to construct an $(1 + \sqrt[3]{3})^n \cdot n^{\mathcal{O}(1)}$ time graph coloring algorithm, which was the fastest one known for twenty-five years. In 2003 Eppstein [5] reduced the running time of a graph coloring to $\mathcal{O}(2.4151^n)$. In 2006 the running time was reduced [1, 12] to $\mathcal{O}(2^n)$. For an overview of the field, see [7].

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Fomin et al. [6] constructed an algorithm for listing all minimal dominating sets of a graph on n vertices in time $\mathcal{O}(1.7159^n)$. There were also given graphs ($n/6$ disjoint copies of the octahedron) having $15^{n/6} \approx 1.5704^n$ minimal dominating sets. This establishes a lower bound on the running time of an algorithm for listing all minimal dominating sets of a given graph.

The number of maximal independent sets in a tree was investigated in [16]. Couturier et al. [4] considered minimal dominating sets in various classes of graphs. The authors of [11] investigated the enumeration of minimal dominating sets in graphs.

Bród and Skupień [2] gave bounds on the number of dominating sets of a tree. They also characterized the extremal trees. The authors of [3] investigated the number of minimal dominating sets in trees containing all leaves.

Skupień [15] conjectured that every tree of order n has at most $2^{n/2}$ minimal dominating sets. It turns out that there are trees having more than $2^{n/2}$ minimal dominating sets, which contradicts the conjecture. We construct a family of trees of both parities of the order for which the number of minimal dominating sets exceeds 1.4167^n, thus exceeding $2^{n/2}$. Since $2^{n/2}$ is not a correct upper bound on the number of minimal dominating sets of a tree, we aim to prove a correct one. We provide an algorithm for listing all minimal dominating sets of a tree of order n in time $\mathcal{O}(1.4656^n)$. This implies that every tree has at most 1.4656^n minimal dominating sets.

2 Disproof of the conjecture

Now we give an infinite family of trees $\{T_k\}_{k=1}^{\infty}$ of odd and even order for which the number of minimal dominating sets exceeds $1.4167^n > 2^{n/2}$. Let T_1 be the tree given in Figure 1. Let T_{k+1} be a tree obtained from T_k by adding an edge connecting one of its support vertices to a support vertex of T_1.

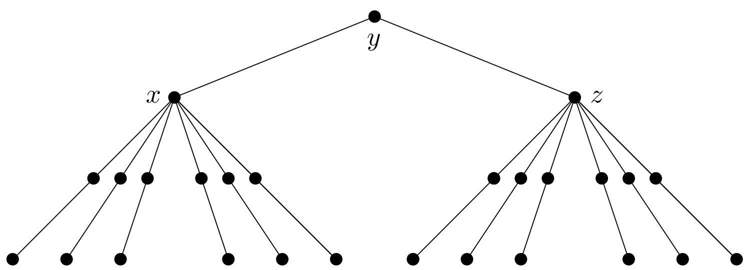

Figure 1: The tree T_1 with 27 vertices

Now we calculate the number of minimal dominating sets of a tree T_k. We root the tree T_1 at the vertex y. Let D be a minimal dominating set of T_1. Observe that for every leaf, either it belongs to the set D or its neighbor belongs to D. First assume that some child of x belongs to the set D, and also some child of z belongs to D. Thus both vertices x and z are already dominated. Since the vertex y has to be dominated, we have either $x \in D$ or $y \in D$ or $z \in D$. There are $3 \cdot (2^6 - 1)^2$ such minimal dominating sets. Now assume that some child of x belongs to the set D, while no child of z belongs to D. We have either $y \in D$ or $z \in D$ as the vertices y and z have to be dominated. There are $2 \cdot (2^6 - 1)$ such minimal dominating sets. The possibility when some child of z belongs to the set D and no child of x belongs to D is similar. If no child of x and z belongs to the set D, then either $x, z \in D$ or $y \in D$. Now we conclude that the tree T_1 has $3 \cdot (2^6 - 1)^2 + 2 \cdot 2 \cdot (2^6 - 1) + 2 = 12161$ minimal dominating sets. While constructing trees of the family $\{T_k\}_{k=1}^{\infty}$, support vertices are connected by edges. Let us observe that every support vertex is already dominated, as it is adjacent to a leaf. This implies that the new edges do not influence the number of minimal dominating sets. Consequently, the tree T_k has 12161^k minimal dominating sets. We have $\sqrt[27k]{12161^k} = \sqrt[27]{12161} \approx 1.41676 > 1.4167$. This implies that $\Omega(1.4167^n)$ is a lower bound on the running time of an algorithm for listing all minimal dominating sets of a given tree of order n.

Independently, in 2011 Górska [8] found an infinite family of T_1-like trees, in which the vertices x and z can have arbitrarily equitably many children. This left the conjecture open only for trees of even order.

3 Listing algorithm

In this section we describe an algorithm, which lists all minimal dominating sets of a given input tree T. Denote by $\mathcal{F}(T)$ the family of sets returned by the algorithm.

Algorithm

Let T be a tree. If $\mathrm{diam}(T) = 0$, then $T = P_1 = v_1$. Let $\mathcal{F}(T) = \{\{v_1\}\}$. If $\mathrm{diam}(T) = 1$, then $T = P_2 = v_1v_2$. Let $\mathcal{F}(T) = \{\{v_1\}, \{v_2\}\}$. If $\mathrm{diam}(T) = 2$, then T is a star. Denote by x the support vertex of T. Let $\mathcal{F}(T) = \{\{x\}, V(T) \setminus \{x\}\}$.

Now assume that $\mathrm{diam}(T) \geq 3$. First assume that some support vertex of T, say x, is strong. Let y be a leaf adjacent to x. Let $T' = T - y$, and let

$$\mathcal{F}(T) = \{D' \colon x \in D' \in \mathcal{F}(T')\} \cup \{D' \cup \{y\} \colon x \notin D' \in \mathcal{F}(T')\}.$$

Now assume that every support vertex of T is weak. We root T at a vertex r of maximum eccentricity $\mathrm{diam}(T)$. Let t be a leaf at maximum distance from r, v be the parent of t, and u be the parent of v in the rooted tree. If $\mathrm{diam}(T) \geq 4$, then let w be the parent of u. Denote by T_x the subtree induced by a vertex x and its descendants in the rooted tree T.

Assume that some child of u, say x, is a leaf. Let $T' = T - T_v$, and let

$$\mathcal{F}(T) = \{D' \cup \{v\}, D' \cup \{t\} \colon D' \in \mathcal{F}(T')\}.$$

Now assume that every child of u is a support vertex. The children of u are denoted by $k_1, k_2, \ldots, k_{d_T(u)-1}$, where $k_1 = v$. Let l_i mean the leaf adjacent to k_i. Let $p_i \in \{k_i, l_i\}$. Denote by w the parent of u. The neighbors of w other than u we denote by $m_1, m_2, \ldots, m_{d_T(w)-1}$. Let $T' = T - T_{k_1} - T_{k_2} - \ldots - T_{k_{d_T(u)-1}}$ and $T'' = T - T_u$. The components of $T'' - w$ are denoted by $T_1, T_2, \ldots, T_{d_T(w)-1}$, where $m_i \in V(T_i)$.

Let $\mathcal{F}(T)$ be a family as follows,

$$\begin{aligned} & \left\{ D' \cup \bigcup_{1 \leq i \leq d_T(u)-1} \{l_i\} \colon D' \in \mathcal{F}(T') \right\} \\ \cup & \left\{ D'' \cup \bigcup_{1 \leq i \leq d_T(u)-1} \{p_i\} \colon D'' \in \mathcal{F}(T'') \text{ and } \exists_j \; p_j = k_j \right\} \\ \cup & \left\{ \bigcup_{1 \leq i \leq d_T(w)-1} D_i \cup \{u\} \cup \bigcup_{1 \leq j \leq d_T(u)-1} \{p_j\} \colon m_j \notin D_j \in \mathcal{F}(T_j) \text{ and } \exists_t \; p_t = k_t \right\}, \end{aligned}$$

where the third component is ignored if w is adjacent to a leaf.

4 Bounding the number of minimal dominating sets

Now we prove that the running time of the algorithm from the previous section is $\mathcal{O}(1.4656^n)$.

Theorem 1 *For every tree T of order n, the algorithm from the previous section lists all minimal dominating sets in time $\mathcal{O}(1.4656^n)$.*

Proof. We prove that the running time of the algorithm is $\mathcal{O}(1.4656^n)$. Moreover, we prove that the number of minimal dominating sets is at most α^n, where $\alpha \approx 1.46557 < 1.4656$ is the positive solution of the equation $x^4 - x^2 - x - 1 = 0$.

We proceed by induction on the number n of vertices of a tree T. If $\text{diam}(T) = 0$, then $T = P_1 = v_1$. Obviously, $\{v_1\}$ is the only minimal dominating set of the path P_1. We have $n = 1$ and $|\mathcal{F}(T)| = 1$. Obviously, $1 < \alpha$. If $\text{diam}(T) = 1$, then $T = P_2 = v_1v_2$. It is easy to see that $\{v_1\}$ and $\{v_2\}$ are the only two minimal dominating sets of the path P_2. We have $n = 2$ and $|\mathcal{F}(T)| = 2$. We also have $2 < \alpha^2$ as $\alpha > \sqrt{2}$. If $\text{diam}(T) = 2$, then T is a star. Denote by x the support vertex of T. It is easy to observe that $\{x\}$ and $V(T) \setminus \{x\}$ are the only two minimal dominating sets of the tree T. We have $n \geq 3$ and $|\mathcal{F}(T)| = 2$. Consequently, $2 < \alpha^2 < \alpha^n$.

Now assume that $\text{diam}(T) \geq 3$. First assume that some support vertex of T, say x, is strong. Let y be a leaf adjacent to x. Let $T' = T - y$. Let D' be a minimal dominating set of the tree T'. If $x \in D'$, then it is easy to see that D' is a minimal dominating set of the tree T. Now assume that $x \notin D'$. It is easy to observe that $D' \cup \{y\}$ is a minimal dominating set of the tree T. Thus all elements of the family $\mathcal{F}(T)$ are minimal dominating sets of the tree T. Now let D be any minimal dominating set of the tree T. Clearly, either the vertex x belongs to the set D or all leaves adjacent to x belong to the set D. If $x \in D$, then it is easy to see that D is a minimal dominating set of the tree T'. By the inductive hypothesis we have $D \in \mathcal{F}(T')$. Now assume that $x \notin D$. It is evident that $D \setminus \{y\}$ is a minimal dominating set of the tree T'. By the inductive hypothesis we have $D \setminus \{y\} \in \mathcal{F}(T')$. Therefore the family $\mathcal{F}(T)$ contains all minimal dominating sets of the tree T. Now we get $|\mathcal{F}(T)| = |\{D' \colon x \in D' \in \mathcal{F}(T')\}| + |\{D' \cup \{y\} \colon x \notin D' \in \mathcal{F}(T')\}| = |\mathcal{F}(T')| \leq \alpha^{n-1} < \alpha^n$. Henceforth, we can assume that every support vertex of T is weak.

We now root T at a vertex r of maximum eccentricity $\text{diam}(T)$. Let t be a leaf at maximum distance from r, v be the parent of t, and u be the parent of v in the rooted tree. If $\text{diam}(T) \geq 4$, then let w be the parent of u. Denote by T_x the subtree induced by a vertex x and its descendants in the rooted tree T.

Assume that some child of u, say x, is a leaf. Let $T' = T - T_v$. Let us observe that all elements of the family $\mathcal{F}(T)$ are minimal dominating sets of the tree T. Now let D be any minimal dominating set of the tree T. We have either $v \in D$ or $t \in D$ as the vertex t has to be dominated and the set D is minimal. Similarly, either $u \in D$ or $x \in D$. If $t \in D$, then observe that $D \setminus \{t\}$ is a minimal dominating set of the tree T'. By the inductive hypothesis we have $D \setminus \{t\} \in \mathcal{F}(T')$. Now assume that $v \in D$. Let us observe that $D \setminus \{v\}$ is a minimal dominating set of the tree T' as the vertex u is still dominated. By the inductive hypothesis we have $D \setminus \{v\} \in \mathcal{F}(T')$. Therefore the family $\mathcal{F}(T)$ contains all minimal dominating sets of the tree T. Now we get $|\mathcal{F}(T)| = 2|\mathcal{F}(T')| \leq 2 \cdot \alpha^{n-2} < \alpha^2 \cdot \alpha^{n-2} = \alpha^n$.

Now assume that every child of u is a support vertex. We use the same notation as in the description of the algorithm. Let $T' = T - T_{k_1} - T_{k_2} - \ldots - T_{k_{d_T(u)-1}}$ and $T'' = T - T_u$. The components of $T'' - w$ we denote by $T_1, T_2, \ldots, T_{d_T(w)-1}$, where $m_i \in V(T_i)$. It is not very difficult to verify that all elements of the family $\mathcal{F}(T)$ are minimal dominating sets of the tree T. Now let D be any minimal dominating set of the tree T. If all leaves of T_u belong to the set D, then observe that $D \setminus \{l_1, l_2, \ldots, l_{d_T(u)-1}\}$ is a minimal dominating set of the tree T'. Now assume that some support vertex of T_u belongs to the set D. If $u \notin D$, then observe that $D \cap V(T'')$ is a minimal dominating set of the tree T''. Now assume that $u \in D$. Let us observe that neither w nor any of its neighbors other than u belongs to the set D, otherwise $D \setminus \{u\}$ is a dominating set of the tree T, a contradiction to the minimality of D. Let us observe that $D \cap V(T_i)$ is a minimal dominating set of the tree T_i, which does not contain the vertex m_i. Now we get

$$\begin{aligned}
|\mathcal{F}(T)| &\leq |\mathcal{F}(T')| + (2^{d_T(u)-1} - 1)\left(|\mathcal{F}(T'')| + \prod_{1 \leq i \leq d_T(w)-1} |D_i \in \mathcal{F}(T_i)\colon m_i \notin D_i|\right) \\
&< |\mathcal{F}(T')| + (2^{d_T(u)-1} - 1)\left(|\mathcal{F}(T'')| + \prod_{1 \leq i \leq d_T(w)-1} |\mathcal{F}(T_i)|\right) \\
&\leq \alpha^{n-2d_T(u)+2} + (2^{d_T(u)-1} - 1)(\alpha^{n-2d_T(u)+1} + \alpha^{n-2d_T(u)}).
\end{aligned}$$

To show that $\alpha^{n-2d_T(u)+2} + (2^{d_T(u)-1} - 1)(\alpha^{n-2d_T(u)+1} + \alpha^{n-2d_T(u)}) \leq \alpha^n$, it suffices to show that $\alpha^2 + (2^{d_T(u)-1} - 1)(\alpha + 1) \leq \alpha^{2d_T(u)}$. We prove this by induction on the degree of the vertex u. For $d_T(u) = 2$ we have $\alpha^2 + (2^{d_T(u)-1} - 1)(\alpha + 1) = \alpha^2 + \alpha + 1 = \alpha^4 = \alpha^{2d_T(u)}$. Now we prove that if the inequality $\alpha^2 + (2^{d_T(u)-1} - 1)(\alpha + 1) \leq \alpha^{2d_T(u)}$ is satisfied for an integer $k = d_T(u) \geq 2$, then it is also satisfied for $k + 1$. We have $\alpha^2 + (2^k - 1)(\alpha + 1) = \alpha^4 - \alpha - 1 + (2^k - 1)(\alpha + 1) = \alpha^4 + (2^k - 2)(\alpha + 1) = \alpha^4$ $+2(2^{k-1} - 1)(\alpha+1) = \alpha^4 - 2\alpha^2 + 2[\alpha^2 + (2^{k-1} - 1)(\alpha+1)] \leq \alpha^4 - 2\alpha^2 + 2\alpha^{2k} = \alpha^4 - 2\alpha^2 + 2\alpha^{2k} + \alpha^{2k+2} - \alpha^{2k+2}$ $= \alpha^{2k+2} + \alpha^2(\alpha^2 - 2) + \alpha^{2k}(2 - \alpha^2) = \alpha^{2k+2} + (\alpha^2 - 2)(\alpha^2 - \alpha^{2k}) < \alpha^{2k+2}$. ■

It follows from the proof of the previous theorem that any tree of order n has less than 1.4656^n minimal dominating sets.

Corollary 2 *Every tree of order n has at most α^n minimal dominating sets, where $\alpha \approx 1.46557$ is the positive solution of the equation $x^4 - x^2 - x - 1 = 0$.*

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