Bipartite theory of graphs: outer-independent domination

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Abstract

Let $G = (V, E)$ be a bipartite graph with partite sets X and Y. Two vertices of X are X-adjacent if they have a common neighbor in Y, and they are X-independent otherwise. A subset $D \subseteq X$ is an X-outer-independent dominating set of G if every vertex of $X \setminus D$ has an X-neighbor in D, and all vertices of $X \setminus D$ are pairwise X-independent. The X-outer-independent domination number of G, denoted by $\gamma_X^{oi}(G)$, is the minimum cardinality of an X-outer-independent dominating set of G. We prove several properties and bounds on the number $\gamma_X^{oi}(G)$.

Keywords: X-dominating set; Y-dominating set; X-independent set, X-outer-independent dominating set.

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Given a graph G, we can construct a bipartite graph G^* which represents G. Similarly, for any problem, say P, on an arbitrary graph G, there is a corresponding problem Q on a bipartite graph G^* such that a solution for Q provides a solution for P. The so called bipartite theory of graphs was introduced by Hedetniemi and Laskar [1, 2]. They also defined the concepts of X-domination and Y-domination. Bipartite domination was further studied for example in [5, 6]. We initiate the study of X-outer-independent domination.

Let $G = (V, E)$ be a graph. The number of vertices of G we denote by n, thus $|V(G)| = n$. By the neighborhood of a vertex v of G we mean the set $N_G(v) = \{u \in V(G) \colon uv \in E(G)\}$. We say that a vertex is isolated if it has no neighbors, while it is universal if it is adjacent to all other vertices. The degree of a vertex v, denoted by $d_G(v)$, is the cardinality of its neighborhood. The minimum degree among all vertices of G we denote by $\delta(G)$. Let uv be an edge of G. By subdividing the edge uv we mean removing it, and adding a new vertex, say x, along with two new edges ux and xv. We say that a subset of $V(G)$ is independent if there is no edge between any two vertices of this set. The independence number of a graph G, denoted by $\alpha(G)$, is the maximum cardinality of an independent subset of the set of vertices of G. The clique number of G, denoted by $\omega(G)$, is the number of vertices of a greatest complete graph which is a subgraph of G.

A subset $D \subseteq V(G)$ is a dominating set of G if every vertex of $V(G) \setminus D$ has a neighbor in D. The domination number of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. For a comprehensive survey of domination in graphs, see [3, 4].

A graph $G = (V, E)$ is bipartite if its set of vertices can be partitioned into two subsets X and Y such that for every edge $uv \in E$, either $u \in X$ and $v \in Y$, or $u \in Y$ and $v \in X$ (that is, every edge joins a vertex of X with a vertex of Y, or equivalently, no edge joins two vertices of X or two vertices of Y).

Let $G = (X, Y, E)$ denote a bipartite graph with partite sets X and Y. Let G^* denote the graph obtained from G by removing all leaves and isolated vertices of Y. Let $Y^* = Y \cap V(G^*)$.

For bipartite graphs G without isolated vertices in Y, a subset D of X is a Y-dominating set if every vertex of Y has a neighbor in D. The minimum cardinality of a Y-dominating set of G is called the Y-domination number of G, and is denoted by $\gamma_Y(G)$.

We say that two vertices of X are X-adjacent if they have a common neighbor in Y. By the X-neighborhood of a vertex v of X we mean the set $N_X(v) = \{u \in X \colon u \text{ and } v \text{ are } X\text{-adjacent}\}$. The X-degree of v, denoted by $d_X(v)$, is the cardinality of the set $N_X(v)$. The minimum X-degree is denoted by $\delta_X(G)$.

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A subset $D \subseteq X$ is an X-dominating set of G if every vertex of $X \setminus D$ has an X-neighbor in D. The minimum cardinality of an X-dominating set of G is called the X-domination number of G, and is denoted by $\gamma_X(G)$.

A subset D of X is an X-independent set if no two vertices of D are X-adjacent. The maximum cardinality of an X-independent set of G is called the X-independence number of G, and is denoted by $\alpha_X(G)$.

A subset D of X is called an X-clique if every two vertices of D are X-adjacent. The maximum cardinality of an X-clique in G is called the X-clique number of G, and is denoted by $\omega_X(G)$.

Let $D \subseteq X$ and let $v \in D$. A vertex $u \in X \setminus D$ is called a private X-neighbor of v with respect to D if v is the only X-neighbor of u in D.

A subset $D \subseteq X$ is an X-outer-independent dominating set, abbreviated XOIDS, of G if every vertex of $X \setminus D$ has an X-neighbor in D, and the set $X \setminus D$ is X-independent. The minimum cardinality of an X-outer-independent dominating set of G is called the X-outer-independent domination number of G, and is denoted by $\gamma_X^{oi}(G)$.

We begin with the following three observations.

Observation 1 *In any bipartite graph without isolated vertices in X, every Y-dominating set is an X-dominating set.*

Observation 2 *Let G be a bipartite graph without leaves and isolated vertices in Y. If S is an X-independent set of G, then $X \setminus S$ is a Y-dominating set of G.*

Observation 3 *For every bipartite graph G without leaves and isolated vertices in Y we have $\gamma_X^{oi}(G) \geq \gamma_Y(G)$.*

We now give a necessary and sufficient condition for that the X-outer-independent domination and the Y-domination numbers of a bipartite graph are equal.

Proposition 4 *For a bipartite graph G we have $\gamma_X^{oi}(G) = \gamma_Y(G)$ if and only if there exists a $\gamma_Y(G)$-set D such that $X \setminus D$ is X-independent.*

Proof. Let D be a $\gamma_Y(G)$-set such that $X \setminus D$ is X-independent. Observation 1 implies that D is an X-dominating set of G. Therefore D is a XOIDS of the graph G, and consequently, $\gamma_X^{oi}(G) \leq |D| = \gamma_Y(G)$. On the other hand, by Observation 3 we have $\gamma_X^{oi}(G) \geq \gamma_Y(G)$.

Now assume that for some bipartite graph G we have $\gamma_X^{oi}(G) = \gamma_Y(G)$. Let D be any $\gamma_X^{oi}(G)$-set. Since $X \setminus D$ is X-independent, Observation 2 implies that D is a Y-dominating set of G. We have $|D| = \gamma_X^{oi}(G) = \gamma_Y(G)$. Thus D is a $\gamma_Y(G)$-set. ■

We now show that the X-outer-independent domination number of any bipartite graph is at least its X-clique number minus one.

Proposition 5 *For every bipartite graph G we have $\gamma_X^{oi}(G) \geq \omega_X(G) - 1$.*

Proof. Let D be a $\gamma_X^{oi}(G)$-set. Let A be an X-clique of G of cardinality $\omega_X(G)$. Observe that at most one vertex of A does not belong to D, as the set $X \setminus D$ is X-independent. Therefore $\gamma_X^{oi}(G) \geq |A| - 1$. We now get $\gamma_X^{oi}(G) \geq |A| - 1 = \omega_X(G) - 1$. ■

We now prove that the X-outer-independent domination number of a bipartite graph is not less than the minimum X-degree.

Proposition 6 *For every graph G we have $\gamma_X^{oi}(G) \geq \delta_X(G)$.*

Proof. Let D be a $\gamma_X^{oi}(G)$-set. If $D = X$, then obviously the result is true. Now assume that $D \neq X$. Let x be a vertex of $X \setminus D$. Since the set $X \setminus D$ is X-independent, all vertices, which are X-adjacent to x, belong to the set D. Therefore $|D| \geq d_X(x)$. We now get $\gamma_X^{oi}(G) = |D| \geq d_X(x) \geq \delta_X(G)$. ■

Observation 7 *For every bipartite graph G we have $1 \leq \gamma_X^{oi}(G) \leq |X|$.*

We now characterize all bipartite graphs, which attain the lower bound from the previous observation. For this purpose we introduce a family $\mathcal{G}$ of bipartite graphs G such that G^* is connected, and for some vertex $x \in X$, in the graph $G^* - x$ all vertices of Y^* are leaves.

Theorem 8 *Let G be a bipartite graph. We have $\gamma_X^{oi}(G) = 1$ if and only if $G \in \mathcal{G}$.*

Proof. First, let us observe that $\gamma_X^{oi}(G) = \gamma_X^{oi}(G^*)$. Thus it suffices to prove that $\gamma_X^{oi}(G^*) = 1$ if and only if G belongs to the family $\mathcal{G}$.

If $G \in \mathcal{G}$, then let x be a vertex of X such that in the graph $G^* - x$ all vertices of Y^* are leaves. If $G^* = K_1$, then obviously $\gamma_X^{oi}(G^*) = 1$. Now assume that $G^* \neq K_1$. The graph G^* is connected, thus every vertex of X has a neighbor in Y. Moreover, every vertex of Y^* is adjacent to x in G^* as no vertex of Y^* is a leaf in G^*. This implies that all vertices of $X \setminus \{x\}$ are X-adjacent to x in G^*. All vertices of Y^* are leaves in the graph $G^* - x$, thus no two vertices of X are X-adjacent, that is, the set $X \setminus \{x\}$ is X-independent. We now conclude that $\{x\}$ is a XOIDS of G^*, implying that $\gamma_X^{oi}(G^*) = 1$.

Now assume that $\gamma_X^{oi}(G^*) = 1$. Let $\{x\}$ be a $\gamma_X^{oi}(G^*)$-set. The vertex x is X-adjacent to all other vertices of X, thus the graph G^* is connected. Since $\{x\}$ is a XOIDS, the set $X \setminus \{x\}$ is X-independent. This implies that in the graph $G^* - x$ all vertices of Y^* are leaves. Therefore $G \in \mathcal{G}$. ■

We now characterize all bipartite graphs for which the X-outer-independent domination number equals the cardinality of X.

Theorem 9 *Let G be a bipartite graph. We have $\gamma_X^{oi}(G) = |X|$ if and only if no two vertices of X are X-adjacent.*

Proof. If the X-neighborhoods of all vertices of X are empty, then obviously $\gamma_X^{oi}(G) = |X|$. Now assume that some vertex of X, say x, has an X-neighbor. Observe that $X \setminus \{x\}$ is a XOIDS of the graph G. Therefore $\gamma_X^{oi}(G) < |X|$. ■

We now characterize all connected bipartite graphs for which the X-outer-independent domination number is one less than the cardinality of X.

Theorem 10 *Let G be a connected bipartite graph with $|X| \geq 2$. We have $\gamma_X^{oi}(G) = |X| - 1$ if and only if every two vertices of X are X-adjacent.*

Proof. If all vertices of X are pairwise X-adjacent, then let x be one of them. Observe that $X \setminus \{x\}$ is a XOIDS of the graph G. Thus $\gamma_X^{oi}(G) \leq |X| - 1$. On the other hand, using Proposition 5 we get $\gamma_X^{oi}(G) \geq \omega_X(G) - 1 = |X| - 1$.

If some two vertices of X, say v_1 and v_2, are not X-adjacent, then let us observe that $X \setminus \{v_1, v_2\}$ is a XOIDS of the graph G. Therefore $\gamma_X^{oi}(G) \leq |X| - 2 < |X| - 1$. ■

Corollary 11 *For every integer $p \geq 2$ we have $\gamma_X^{oi}(K_{p,q}) = p - 1$.*

We now give a lower bound on the X-outer-independent domination number of any bipartite graph.

Theorem 12 *For every bipartite graph G with $|X| = p$, $|Y| = q$ and $|E| = m$ we have $\gamma_X^{oi}(G) \geq p - 1/2 - \sqrt{pq - m + 1/4}$.*

Proof. Let D be a $\gamma_X^{oi}(G)$-set. The definition of an X-outer-independent dominating set implies that for every $x \in X \setminus D$ there exists $y \in Y$ such that $N_G(y) \subseteq D \cup \{x\}$. Thus each vertex of $X \setminus D$ is not adjacent to $p - \gamma_X^{oi}(G) - 1$ vertices of Y. We now get

$$\begin{aligned} m &\leq pq - (p - \gamma_X^{oi}(G)) \cdot (p - \gamma_X^{oi}(G) - 1) \\ &= pq - ((p - \gamma_X^{oi}(G))^2 - (p - \gamma_X^{oi}(G))) \\ &= pq + 1/4 - ((p - \gamma_X^{oi}(G))^2 - (p - \gamma_X^{oi}(G)) + 1/4) \\ &= pq + 1/4 - (p - \gamma_X^{oi}(G) - 1/2)^2. \end{aligned}$$

This implies that

$$p - \gamma_X^{oi}(G) - \frac{1}{2} \leq \sqrt{pq - m + \frac{1}{4}},$$

and consequently,

$$\gamma_X^{oi}(G) \geq p - \frac{1}{2} - \sqrt{pq - m + \frac{1}{4}}.$$

■

Let us observe that the bound from the previous theorem is tight. For a complete bipartite graph $K_{p,q}$ with $p \geq 2$ we have $p - 1/2 - \sqrt{pq - m + 1/4} = p - 1/2 - \sqrt{pq - pq + 1/4} = p - 1 = \gamma_X^{oi}(K_{p,q})$.

We have the following characterization of X-outer-independent dominating sets, which are minimal.

Theorem 13 *Let D be a XOIDS of a bipartite graph G. Then D is minimal if and only if for every vertex of D, say u, at least one of the following conditions is satisfied:*

(i) there is a vertex $v \in X \setminus D$ such that v is a private X-neighbor of u with respect to D;

(ii) $(X \setminus D) \cup \{u\}$ is not an X-independent set of G.

Proof. If D is minimal, then $D \setminus \{u\}$ is not a XOIDS of G. Thus $D \setminus \{u\}$ is not an X-dominating set of G or $(X \setminus D) \cup \{u\}$ is not an X-independent set of G. If $D \setminus \{u\}$ is not an X-dominating set of G, then there is a vertex $v \in X \setminus (D \setminus \{u\})$, which is not X-adjacent to any vertex of $D \setminus \{u\}$, but is X-adjacent to a vertex of D, namely u. Therefore we get (i). If $(X \setminus D) \cup \{u\}$ is not an X-independent set of G, then we get (ii).

Now assume that at least one of the conditions (i) and (ii) holds, and suppose that D is not minimal. Then there exists a vertex $u \in D$ such that $D \setminus \{u\}$ is a XOIDS of G. Equivalently, $D \setminus \{u\}$ is an X-dominating set of G, and the set $(X \setminus D) \cup \{u\}$ is X-independent. Thus u has no private X-neighbors with respect to D, so the condition (i) does not hold. Since $(X \setminus D) \cup \{u\}$ is an X-independent set of G, we get a contradiction to (ii). ■

We now define the complement of a bipartite graph G, denoted by $\bar{G} = (X, Y, E^1)$, as follows:

(i) no two vertices in X are adjacent;

(ii) no two vertices in Y are adjacent;

(iii) $x \in X$ and $y \in Y$ are adjacent in $\bar{G}$ if and only if they are not adjacent in G.

We shall now characterize the graphs G for which $\gamma_X^{oi}(G) + \gamma_X^{oi}(\bar{G}) = 2p - 2$. For this purpose we define a family $\mathcal{F}$ of bipartite graphs such that $|X| \geq 2$, all vertices of X are pairwise X-adjacent, and for any two vertices $u, v \in X$ there is a vertex $y \in Y$ such that neither u nor v is adjacent to y.

Theorem 14 *Let G be a connected bipartite graph such that its complement $\bar{G}$ is also connected. We have $\gamma_X^{oi}(G) + \gamma_X^{oi}(\bar{G}) = 2p - 2$ if and only if $G \in \mathcal{F}$.*

Proof. If $\gamma_X^{oi}(G) + \gamma_X^{oi}(\bar{G}) = 2p - 2$, then observe that $\gamma_X^{oi}(G) = \gamma_X^{oi}(\bar{G}) = p - 1$ as the graphs G and $\bar{G}$ are connected. Theorem 10 implies that all vertices of X are pairwise X-adjacent in both graphs G and $\bar{G}$. Let $u, v \in X$. Since every two vertices of X are pairwise X-adjacent in $\bar{G}$, there is $y \in Y$ such that u and v are adjacent to y in $\bar{G}$, so they are not adjacent to y in G. We now conclude that $G \in \mathcal{F}$. The converse is straightforward. ■

Given an arbitrary graph $G = (V, E)$, we can construct a bipartite graph $VE(G)$ in a way given in [1, 2].

The bipartite graph $VE(G)$: For any graph $G = (V, E)$, let $VE(G)$ be a bipartite graph (V, E, F), where $E = E(G)$ and $F = \{ue \colon e = uv \in E(G)\}$. The graph $VE(G)$ is isomorphic to the graph $S(G)$ (the subdivision graph of G), that is, the graph obtained from G by subdividing each one of its edges exactly once.

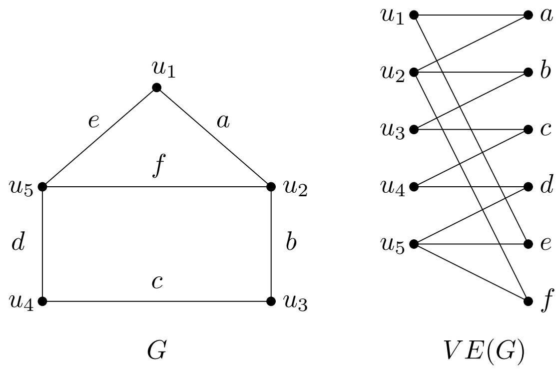

Figure 1: A graph G and its bipartite construction $VE(G)$

We now define the outer-independent domination number of a graph.

A subset D of $V(G)$ is an outer-independent dominating set of G if every vertex of $V(G) \setminus D$ has an X-neighbor in D, and the set $V(G) \setminus D$ is is independent. The outer-independent domination number of a graph G, denoted by $\gamma^{oi}(G)$, is the minimum cardinality of an outer-independent dominating set of G.

Theorem 15 *For every graph G we have $\gamma^{oi}(G) = \gamma_X^{oi}(VE(G)) = |V(G)| - \alpha(G)$.*

Proof. Let D be a $\gamma_X^{oi}(VE(G))$-set, where $VE(G) = (X, Y, E)$. The set $X \setminus D$ is X-independent and every vertex of $X \setminus D$ has an X-neighbor in D. Equivalently, in the graph G, the set $V(G) \setminus D$ is independent and every vertex of $V(G) \setminus D$ has a neighbor in D. Therefore D is an outer-independent dominating set of G. We now get $\gamma^{oi}(G) \leq |D| = \gamma_X^{oi}(VE(G))$.

Now let S be a $\gamma^{oi}(G)$-set. Thus $V(G) \setminus S$ is an independent set, and every vertex of $V(G) \setminus S$ has a neighbor in S. This implies that in the graph $VE(G)$, the set $X \setminus S$ is X-independent and every vertex of $X \setminus S$ has an X-neighbor in S. Thus S is an X-outer-independent dominating set of the graph $VE(G)$. Hence, $\gamma_X^{oi}(VE(G)) \leq |S| = \gamma^{oi}(G)$.

We now conclude that $\gamma^{oi}(G) = \gamma_X^{oi}(VE(G))$. It is not very difficult to obtain the equality $\gamma^{oi}(G) = |V(G)| - \alpha(G)$. ■

A subset D of $E(G)$ is called an edge outer-independent dominating set of G if for every edge $e \in E \setminus D$ there is an edge $f \in D$ such that e and f are adjacent, and the set $E \setminus D$ is edge-independent. The edge outer-independent domination number of a graph G, denoted by $\gamma_e^{oi}(G)$, is the minimum cardinality of an edge outer-independent dominating set of G.

Similarly as for the outer-independent domination, we obtain the following result.

Theorem 16 *Let G be a graph, and let H be a bipartite graph obtained from $EV(G)$ by swapping its partite sets. We have $\gamma_e^{oi}(G) = \gamma_X^{oi}(H)$.*

The advantage of the concept of X-outer-independent dominating set is that, if it is applied to the vertex set V of the bipartite graph VE, then we get an outer-independent dominating set, and when applied to the edge set E, we get an edge outer-independent dominating set. Thus with a single parameter X-outer-independent domination number, we can study the outer-independent domination and the edge outer-independent domination numbers of a graph.

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