On homogeneously representable interval graphs

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Abstract

We determine all graphs whose line graphs (middle graphs, total graphs, respectively) are homogeneously representable interval graphs.

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A graph $G = (V, E)$ is said to be an interval graph if it is possible to assign to each vertex of G a closed interval on the real line such that two distinct vertices of G are adjacent if and only if the corresponding intervals have a non-empty intersection, that is, if there exists a collection $\mathcal{I} = \{I_v : v \in V(G)\}$ of closed intervals on the real line such that G is isomorphic to the intersection graph $\Omega(\mathcal{I})$ of $\mathcal{I}$. In such a situation, the collection $\mathcal{I}$ is called an interval representation of G. Without loss of generality we may assume that an interval representation consists of closed, nonempty, finite intervals in which all end points of the intervals are distinct. The first characterization of interval graphs has been proved by Lekkerkerker and Boland [3]. In some applications of interval graphs it is desirable to have an interval graph with as few different interval representations as possible. In [4] a class of interval graphs whose representations are far from being unique is demonstrated.

Let $\mathcal{I} = \{I_1, \ldots, I_p\}$ be a set of intervals of the real line, where $I_i = [a_i, b_i]$ for $i = 1, 2, \ldots, p$. An interval I_i is called an end interval of the set $\mathcal{I}$ if $a_i \leq a_j$ for all j, or $b_i \geq b_j$ for all j. A graph G is called a homogeneously representable interval graph (shortly, an HRI graph) if for every vertex v of G there exists an interval representation of G in which the interval representing v is an end interval. Homogeneously representable interval graphs were characterized in terms of forbidden subgraphs by Skrien and Gimbel [4].

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Theorem 1 (Skrien and Gimbel) *A graph G is an HRI graph if and only if it does not contain any of the graphs P_4, C_4, C_5 or G_1 shown in Figure 1 as an induced subgraph.*

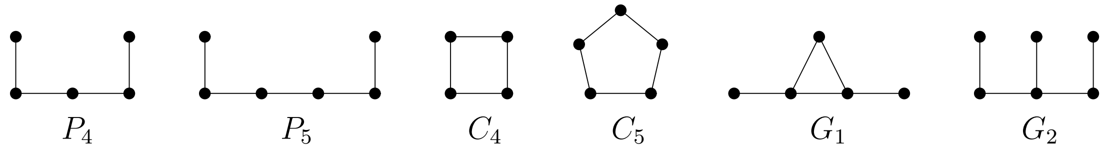

Figure 1

The line graph of a graph G, denoted by $L(G)$, is the intersection graph $\Omega(\overline{E}(G))$ of the family $\overline{E}(G) = \{\{u,v\}\colon uv \in E(G)\}$, that is, $L(G)$ is the graph whose vertices are in one-to-one correspondence with the edges of G, and two vertices of $L(G)$ are adjacent if and only if the corresponding edges of G are adjacent. Whitney [5] proved that $K_{1,3}$, K_3 is the only pair of non-isomorphic connected graphs with isomorphic line graphs. In the next two theorems we characterize all graphs G whose line graphs $L(G)$ are homogeneously representable interval graphs.

Theorem 2 *The line graph $L(G)$ of a graph G is an HRI graph if and only if G contains no P_5, C_4, C_5 or G_2 (shown in Figure 1) as a subgraph.*

Proof. Note that $P_4 = L(P_5)$, $C_4 = L(C_4)$, $C_5 = L(C_5)$, and $G_1 = L(G_2)$. Now, Whitney's theorem implies that if at least one of the graphs P_4, C_4, C_5, and G_1 is an induced subgraph of the line graph $L(G)$, then at least one of the graphs P_5, C_4, C_5, and G_2 is a subgraph of G. From this and from Theorem 1 it follows that if $L(G)$ is not an HRI graph, then at least one of the graphs P_5, C_4, C_5, and G_2 is a subgraph of G. The opposite implication is straightforward. ■

Theorem 3 *The line graph $L(G)$ of a graph G is an HRI graph if and only if every connected component of G is a subgraph of any of the graphs H_1, H_2, and H_3 given in Figure 2.*

Proof. Since $L(G)$ is an HRI graph if and only if every connected component of $L(G)$ is an HRI graph, without loss of generality we may assume that G is connected and different from K_1. First note that if G is a subgraph of any of the graphs given in Figure 2, then it contains no P_5, C_4, C_5 or G_2 as a subgraph, and therefore $L(G)$ is an HRI graph, by Theorem 2.

Now assume that $L(G)$ is an HRI graph. According to Theorem 2, the graph G does not contain P_5, C_4, C_5 or G_2 as a subgraph. Let $P = (v_0, v_1, \ldots, v_d)$ be a longest path in G. Since P_5 is not a subgraph of G and $G \neq K_1$, we have $1 \leq d$

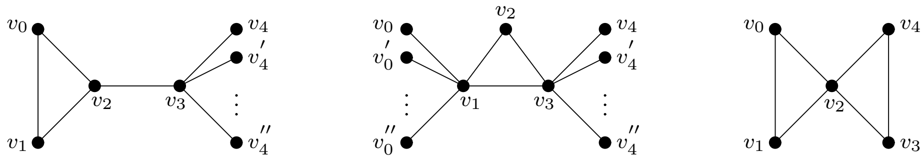

Figure 2

≤ 4. If $d = 1$, then $G = K_2$ and G is a subgraph of H_i. If $d = 2$, then G is a star or a complete graph on three vertices. Notice that G is a subgraph of the graphs H_1 and H_2. If $d = 3$ and P has no chord in G, then it follows from the choice of P that the sets $N_G(v_1)$ and $N_G(v_2)$ are disjoint, and every vertex of $N_G(v_1) \cup N_G(v_2) \setminus \{v_1, v_2\}$ is a leaf in G. Thus G is a double star, and it is a subgraph of H_2. Now assume that $d = 3$ and P has a chord in G. From the absence of C_4 in G, it follows that either v_0v_2 or v_1v_3 is a chord of P in G. Without loss of generality, assume that v_0v_2 is a chord of P in G. Since P is a longest path in G, we have $N_G(v_0) = \{v_1, v_2\}$, $N_G(v_1) = \{v_0, v_2\}$, and each vertex of $N_G(v_2) \setminus \{v_0, v_1\}$ is a leaf in G. Therefore G can be obtained from K_3 by attaching a positive number of leaves to exactly one vertex of K_3. Certainly, G is a subgraph of H_2. Now assume that $d = 4$. From the absence of C_4 and C_5 in G and from the choice of P, it easily follows that $N_G(v_0) \setminus \{v_1\} \subseteq \{v_2\}$ and $N_G(v_4) \setminus \{v_3\} \subseteq \{v_2\}$. In addition, $N_G(v_2) \setminus \{v_1, v_3\} \subseteq \{v_0, v_4\}$ as otherwise G_2 would be a subgraph of G. Again from the choice of P and from the absence of C_4 in G, it follows that $N_G(v_1) = \{v_0, v_2\}$ if v_0v_2 is a chord of P in G. Similarly, $N_G(v_3) = \{v_2, v_4\}$ if v_2v_4 is a chord of P in G. This implies that $G = H_3$ if both v_0v_2 and v_2v_4 are chords of P in G. If v_0v_2 is a chord of P and v_2v_4 is not a chord of P, then the choice of P implies that the vertices belonging to $N_G(v_3)$ are independent, and G is a subgraph of H_1. Similarly, G is a subgraph of H_1 if v_2v_4 is a chord and v_0v_2 is not a chord of P in G. Finally assume that neither v_0v_2 nor v_2v_4 is a chord of P in G. Then from the choice of P and from the absence of C_4 in G, it follows that the sets $N_G(v_1) \setminus \{v_2\}$ and $N_G(v_3) \setminus \{v_2\}$ are disjoint and each of them consists of independent vertices. Therefore G is a subgraph of H_2. $\blacksquare$

The middle graph of a graph G, denoted by $M(G)$, is the intersection graph $\Omega(\mathcal{F})$ of the family $\mathcal{F} = \{\{v\} \colon v \in V(G)\} \cup \{\{v, u\} \colon vu \in E(G)\}$. It is known that $M(G)$ is isomorphic to the line graph $L(G \circ K_1)$ (see [1]), where $G \circ K_1$ is a graph obtained by taking the graph G and $|V(G)|$ copies of K_1 and then joining the i-th vertex of G to the i-th copy of K_1.

The following result follows from Theorems 1 and 2.

Theorem 4 *The middle graph $M(G)$ of a graph G is an HRI graph if and only if every connected component of G is isomorphic to K_1 or K_2.*

Proof. If every component of G is isomorphic to K_1 or K_2, then every component of $M(G)$ is $K_1 = M(K_1)$ or $K_{1,2} = M(K_2)$. Thus by Theorem 1, $M(G)$ is an HRI graph. Now assume that $M(G)$ is an HRI graph. Suppose that G has a component different from K_1 and K_2. Then $K_{1,2}$ is a subgraph of G and therefore $G_2 = K_{1,2} \circ K_1$ is a subgraph of $G \circ K_1$. Consequently, by Theorem 2, the middle graph $M(G) = L(G \circ K_1)$ is not an HRI graph, a contradiction. ■

The total graph of a graph G, denoted by $T(G)$, is the intersection graph $\Omega(F)$ of the family $F = \overline{E}(G) \cup \overline{VE}(G) = \{\{v,u\}\colon vu \in E(G)\} \cup \{\{v\} \cup \{\{v,u\} : u \in N_G(v)\}\colon v \in V(G)\}$, that is, $T(G)$ is the graph for which there exists a one-to-one correspondence between its vertices and the vertices and edges of G such that two vertices of $T(G)$ are adjacent if and only if the corresponding elements in G are adjacent or incident. This concept was originated by Behzad [2]. It is interesting to note that the graphs G and $L(G)$ are induced subgraphs of the total graph $T(G)$.

We now determine all graphs whose total graphs are HRI graphs.

Theorem 5 *The total graph $T(G)$ of a graph G is an HRI graph if and only if every connected component of G is isomorphic to K_1, K_2 or $K_{1,2}$.*

Proof. The sufficiency follows immediately from Theorem 1. Now assume that $T(G)$ is an HRI graph. It is easy to see that if $T(G)$ is an interval graph, then every connected component of G is triangle-free. From this and from the absence of G_1 in $T(G)$ (see Theorem 1) it follows that P_3 is not a subgraph of G. Thus every component of G is isomorphic to one of the graphs K_1, K_2, or $K_{1,2}$. ■

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