On trees with total domination number equal to edge-vertex domination number plus one

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Abstract

An edge $e \in E(G)$ dominates a vertex $v \in V(G)$ if e is incident with v or e is incident with a vertex adjacent to v. An edge-vertex dominating set of a graph G is a set D of edges of G such that every vertex of G is edge-vertex dominated by an edge of D. The edge-vertex domination number of a graph G is the minimum cardinality of an edge-vertex dominating set of G. A subset $D \subseteq V(G)$ is a total dominating set of G if every vertex of G has a neighbor in D. The total domination number of G is the minimum cardinality of a total dominating set of G. We characterize all trees with total domination number equal to edge-vertex domination number plus one.

Keywords: total domination, edge-vertex domination, tree.
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1 Introduction

Let $G = (V, E)$ be a graph. By the neighborhood of a vertex v of G we mean the set $N_G(v) = \{u \in V(G) \colon uv \in E(G)\}$. The degree of a vertex v, denoted by $d_G(v)$, is the cardinality of its neighborhood. By a leaf we mean a vertex of degree one, while a support vertex is a vertex adjacent to a leaf. We say that

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a support vertex is strong (weak, respectively) if it is adjacent to at least two leaves (exactly one leaf, respectively). The edge incident with a leaf is called an end edge. The path on n vertices we denote by P_n. Let T be a tree, and let v be a vertex of T. We say that v is adjacent to a path P_n if there is a neighbor of v, say x, such that one of the components of $T - vx$ is a path P_n containing x as a leaf.

A subset $D \subseteq V(G)$ is a dominating set of G if every vertex of $V(G) \setminus D$ has a neighbor in D. The domination number of a graph G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. A subset $D \subseteq V(G)$ is a total dominating set, abbreviated TDS, of G if every vertex of G has a neighbor in D. The total domination number of a graph G, denoted by $\gamma_t(G)$, is the minimum cardinality of a total dominating set of G. For a comprehensive survey of domination in graphs, see [2].

An edge $e \in E(G)$ dominates a vertex $v \in V(G)$ if e is incident with v or e is incident with a vertex adjacent to v. A subset $D \subseteq E(G)$ is an edge-vertex dominating set, abbreviated EVDS, of a graph G if every vertex of G is edge-vertex dominated by an edge of D. The edge-vertex domination number of a graph G, denoted by $\gamma_{ev}(G)$, is the minimum cardinality of an edge-vertex dominating set of G. Edge-vertex domination in graphs was introduced in [4], and was further studied in [3].

Trees with equal domination and total domination numbers were characterized in [1]. We characterize all trees with total domination number equal to edge-vertex domination number plus one.

2 Results

Since the one-vertex graph does not have a total dominating set or an edge-vertex dominating set, in this paper, we consider only trees on at least two vertices.

We begin with the following three straightforward observations.

Observation 1 *Every support vertex of a graph G is in every TDS of G.*

Observation 2 *For every connected graph G of diameter at least three there exists a $\gamma_t(G)$-set that contains no leaf.*

Observation 3 *For every connected graph G of diameter at least three there exists a $\gamma_{ev}(T)$-set that contains no end edge.*

We now prove that the total domination number of any tree is greater than its edge-vertex domination number.

Lemma 4 *For every tree T we have $\gamma_t(T) > \gamma_{ev}(T)$.*

Proof. If $\mathrm{diam}(T) \leq 3$, then we get $\gamma_t(T) = 2 > 1 = \gamma_{ev}(T)$. Now assume that $\mathrm{diam}(T) \geq 4$. Thus the order n of the tree T is at least five. We prove the result by the induction on the number n. Assume that the theorem is true for every tree T' of order $n' < n$.

First assume that some support vertex of T, say x, is strong. Let y be a leaf adjacent to x. Let $T' = T - y$. Let D' be a $\gamma_{ev}(T')$-set. It is easy to see that D' is an EVDS of the tree T. Thus $\gamma_{ev}(T) \leq \gamma_{ev}(T')$. Obviously, $\gamma_t(T') \leq \gamma_t(T)$. We now get $\gamma_t(T) \geq \gamma_t(T') > \gamma_{ev}(T') \geq \gamma_{ev}(T)$. Henceforth, we can assume that every support vertex of T is weak.

We now root T at a vertex r of maximum eccentricity $\mathrm{diam}(T)$. Let t be a leaf at maximum distance from r, v be the parent of t, u be the parent of v, and w be the parent of u in the rooted tree. By T_x we denote the subtree induced by a vertex x and its descendants in the rooted tree T.

Assume that some child of u, say x, is a leaf. Let $T' = T - x$. Let D' be a $\gamma_{ev}(T')$-set that contains no end edge. The vertex t has to be dominated, thus $uv \in D$. It is easy to see that D' is an EVDS of the tree T. Thus $\gamma_{ev}(T) \leq \gamma_{ev}(T')$. Obviously, $\gamma_t(T') \leq \gamma_t(T)$. We now get $\gamma_t(T) \geq \gamma_t(T') > \gamma_{ev}(T') \geq \gamma_{ev}(T)$.

Now assume that among the children of u there is a support vertex, say x, other than v. Let $T' = T - T_v$. Let D' be a $\gamma_{ev}(T')$-set. It is easy to see that $D' \cup \{uv\}$ is an EVDS of the tree T. Thus $\gamma_{ev}(T) \leq \gamma_{ev}(T') + 1$. Now let D be a $\gamma_t(T)$-set that contains no leaf. By Observation 1 we have $v, x \in D$. Let us observe that $D \setminus \{v\}$ is a TDS of the tree T'. Therefore $\gamma_t(T') \leq \gamma_t(T) - 1$. We now get $\gamma_t(T) \geq \gamma_t(T') + 1 > \gamma_{ev}(T') + 1 \geq \gamma_{ev}(T)$.

Now assume that $d_T(u) = 2$. Let $T' = T - T_u$. Let D' be a $\gamma_{ev}(T')$-set. It is easy to see that $D' \cup \{uv\}$ is an EVDS of the tree T. Thus $\gamma_{ev}(T) \leq \gamma_{ev}(T') + 1$. Now let D be a $\gamma_t(T)$-set that contains no leaf. By Observation 1 we have $v \in D$. The vertex v has to be dominated, thus $u \in D$. Let k be a neighbor of w other than u. If $k \in D$, then $D \setminus \{u, v\}$ is a TDS of the tree T'. Now assume that $k \notin D$. It is easy to observe that $D \cup \{k\} \setminus \{u, v\}$ is a TDS of the tree T'. We now conclude that $\gamma_t(T') \leq \gamma_t(T) - 1$. We get $\gamma_t(T) \geq \gamma_t(T') + 1 > \gamma_{ev}(T') + 1 \geq \gamma_{ev}(T)$. ■

We characterize all trees with total domination number equal to edge-vertex domination number plus one. For this purpose we introduce a family $\mathcal{T}$ of trees $T = T_k$ that can be obtained as follows. Let $T_1 \in \{P_2, P_3\}$. If k is a positive integer, then T_{k+1} can be obtained recursively from T_k by one of the following operations.

- Operation $\mathcal{O}_1$: Attach a vertex by joining it to any support vertex of T_k.
- Operation $\mathcal{O}_2$: Attach a vertex or a path P_2 by joining one of its vertices to a vertex of T_k adjacent to a path P_2.

We now prove that for every tree of the family $\mathcal{T}$, the total domination number is equal to the edge-vertex domination number plus one.

Lemma 5 *If $T \in \mathcal{T}$, then $\gamma_t(T) = \gamma_{ev}(T) + 1$.*

Proof. We use induction on the number k of operations performed to construct the tree T. If $T \in \{P_2, P_3\}$, then obviously $\gamma_t(T) = 2 = \gamma_{ev}(T) + 1$. Let k be a positive integer. Assume that the result is true for every $T' = T_k$ of the family $\mathcal{T}$ constructed by $k-1$ operations. Let $T = T_{k+1}$ be a tree of the family $\mathcal{T}$ constructed by k operations.

First assume that T is obtained from T' by operation $\mathcal{O}_1$. Let D' be a $\gamma_t(T')$-set. It is easy to see that D' is a TDS of the tree T. Thus $\gamma_t(T) \leq \gamma_t(T')$. Obviously, $\gamma_{ev}(T') \leq \gamma_{ev}(T)$. We now get $\gamma_t(T) \leq \gamma_t(T') = \gamma_{ev}(T') + 1 \leq \gamma_{ev}(T) + 1$. On the other hand, by Lemma 4 we have $\gamma_t(T) \geq \gamma_{ev}(T) + 1$. This implies that $\gamma_t(T) = \gamma_{ev}(T) + 1$.

Now assume that T is obtained from T' by operation $\mathcal{O}_2$. The vertex to which is joined a new vertex we denote by x. Let y be a support vertex of degree two adjacent to x. The neighbor of y other than x we denote by z. If we attach a vertex, then let D' be a $\gamma_t(T')$-set that contains no leaf. The vertex y has to be dominated, thus $x \in D'$. It is easy to see that D' is a TDS of the tree T. Thus $\gamma_t(T) \leq \gamma_t(T')$. Obviously, $\gamma_{ev}(T') \leq \gamma_{ev}(T)$. We now get $\gamma_t(T) \leq \gamma_t(T')$ $= \gamma_{ev}(T') + 1 \leq \gamma_{ev}(T)+1$. Now assume that we attach a path P_2, say v_1v_2. Let v_1 be joined to x. Let D' be a $\gamma_t(T')$-set that contains no leaf. The vertex y has to be dominated, thus $x \in D'$. It is easy to observe that $D' \cup \{v_1\}$ is a TDS of the tree T. Thus $\gamma_t(T) \leq \gamma_t(T') + 1$. Now let D be a $\gamma_{ev}(T)$-set that contains no end edge. The vertices z and v_2 have to be dominated, thus $xy, xv_1 \in D$. Let us observe that $D \setminus \{xv_1\}$ is an EVDS of the tree T'. Therefore $\gamma_{ev}(T') \leq \gamma_{ev}(T) - 1$. We now get $\gamma_t(T) \leq \gamma_t(T') + 1 = \gamma_{ev}(T') + 2 \leq \gamma_{ev}(T) + 1$. We conclude that $\gamma_t(T) = \gamma_{ev}(T) + 1$. ∎

We now prove that if the total domination number of a tree is equal to its edge-vertex domination number plus one, then the tree belongs to the family $\mathcal{T}$.

Lemma 6 *Let T be a tree. If $\gamma_t(T) = \gamma_{ev}(T) + 1$, then $T \in \mathcal{T}$.*

Proof. If $\mathrm{diam}(T) = 1$, then $T = P_2 \in \mathcal{T}$. If $\mathrm{diam}(T) = 2$, then T is a star. If $T = P_3$, then $T \in \mathcal{T}$. If T is a star different from P_3, then it can be obtained from P_3 by an appropriate number of operations $\mathcal{O}_1$. Thus $T \in \mathcal{T}$. Now assume that $\mathrm{diam}(T) \geq 3$. Thus the order n of the tree T is at least four. We prove the result by induction on n. Assume that the result is true for every tree T' of order $n' < n$.

First assume that some support vertex of T, say x, is strong. Let y be a leaf adjacent to x. Let $T' = T - y$. Let D' be a $\gamma_{ev}(T')$-set. It is easy to see that D' is an EVDS of the tree T. Thus $\gamma_{ev}(T) \leq \gamma_{ev}(T')$. Obviously, $\gamma_t(T') \leq \gamma_t(T)$. We now get $\gamma_t(T') \leq \gamma_t(T) = \gamma_{ev}(T) + 1 \leq \gamma_{ev}(T') + 1$. On the other hand, by Lemma 4 we have $\gamma_t(T') \geq \gamma_{ev}(T') + 1$. This implies that $\gamma_t(T') = \gamma_{ev}(T') + 1$.

By inductive hypothesis we have $T' \in \mathcal{T}$. The tree T can be obtained from T' by operation $\mathcal{O}_1$. Thus $T \in \mathcal{T}$. Henceforth, we can assume that every support vertex of T is weak.

We now root T at a vertex r of maximum eccentricity $\mathrm{diam}(T)$. Let t be a leaf at maximum distance from r, v be the parent of t, and u be the parent of v in the rooted tree. If $\mathrm{diam}(T) \geq 4$, then let w be the parent of u. If $\mathrm{diam}(T) \geq 5$, then let d be the parent of w. If $\mathrm{diam}(T) \geq 6$, then let e be the parent of d. By T_x we denote the subtree induced by a vertex x and its descendants in the rooted tree T.

Assume that some child of u, say x, is a leaf. Let $T' = T - x$. Let D' be a $\gamma_{ev}(T')$-set that contains no end edge. The vertex t has to be dominated, thus $uv \in D'$. It is easy to see that D' is an EVDS of the tree T. Thus $\gamma_{ev}(T) \leq \gamma_{ev}(T')$. Obviously, $\gamma_t(T') \leq \gamma_t(T)$. We now get $\gamma_t(T') \leq \gamma_t(T) = \gamma_{ev}(T)+1 \leq \gamma_{ev}(T')+1$. This implies that $\gamma_t(T') = \gamma_{ev}(T') + 1$. By inductive hypothesis we have $T' \in \mathcal{T}$. The tree T can be obtained from T' by operation $\mathcal{O}_2$. Thus $T \in \mathcal{T}$.

Now assume that among the children of u there is a support vertex, say x, other than v. Let $T' = T - T_v$. Let D' be a $\gamma_{ev}(T')$-set. It is easy to see that $D' \cup \{uv\}$ is an EVDS of the tree T. Thus $\gamma_{ev}(T) \leq \gamma_{ev}(T') + 1$. Now let D be a $\gamma_t(T)$-set that contains no leaf. By Observation 1 we have $v, x \in D$. Let us observe that $D \setminus \{v\}$ is a TDS of the tree T'. Therefore $\gamma_t(T') \leq \gamma_t(T) - 1$. We now get $\gamma_t(T') \leq \gamma_t(T) - 1 = \gamma_{ev}(T) \leq \gamma_{ev}(T') + 1$. This implies that $\gamma_t(T') = \gamma_{ev}(T') + 1$. By inductive hypothesis we have $T' \in \mathcal{T}$. The tree T can be obtained from T' by operation $\mathcal{O}_2$. Thus $T \in \mathcal{T}$.

Now assume that $d_T(u) = 2$. If $d_T(w) = 1$, then $T = P_4$. Let $T' = P_3 \in \mathcal{T}$. The tree T can be obtained from T' by operation $\mathcal{O}_2$. Thus $T \in \mathcal{T}$. Now assume that $d_T(w) = 2$. If $d_T(d) = 1$, then $T = P_5$. Let $T' = P_3 \in \mathcal{T}$. The tree T can be obtained from T' by operation $\mathcal{O}_2$. Thus $T \in \mathcal{T}$. Now assume that $d_T(d) \geq 2$. Let $T' = T - T_w$. Let D' be a $\gamma_{ev}(T')$-set. It is easy to see that $D' \cup \{uv\}$ is an EVDS of the tree T. Thus $\gamma_{ev}(T) \leq \gamma_{ev}(T') + 1$. Now let us observe that there exists a $\gamma_t(T)$-set that does not contain the vertices t and w. Let D be such a set. By Observation 1 we have $v \in D$. The vertex v has to be dominated, thus $u \in D$. Observe that $D \setminus \{u, v\}$ is a TDS of the tree T'. Therefore $\gamma_t(T') \leq \gamma_t(T) - 2$. We now get $\gamma_t(T') \leq \gamma_t(T) - 2 = \gamma_{ev}(T) - 1 \leq \gamma_{ev}(T') < \gamma_{ev}(T') + 1$.

Now assume that $d_T(w) \geq 3$. First assume that there is a child of w other than u, say k, such that the distance of w to the most distant vertex of T_k is three or two. It suffices to consider only the possibilities when T_k is a path P_3 or P_2, say klm or kl. Let $T' = T - T_u$. Let D' be a $\gamma_{ev}(T')$-set. It is easy to see that $D' \cup \{uv\}$ is an EVDS of the tree T. Thus $\gamma_{ev}(T) \leq \gamma_{ev}(T') + 1$. Now let D be a $\gamma_t(T)$-set that contains no leaf. By Observation 1 we have $v \in D$. The vertices v and l have to be dominated, thus $u, k \in D$. Let us observe that $D \setminus \{u, v\}$ is a TDS of T'. Therefore $\gamma_t(T') \leq \gamma_t(T) - 2$. We now get $\gamma_t(T') \leq \gamma_t(T) - 2 = \gamma_{ev}(T) - 1 \leq \gamma_{ev}(T') < \gamma_{ev}(T') + 1$.

Now assume that some child of w, say x, is a leaf. We can assume that

$d_T(w) = 3$. First assume that some child of d, say k, is a leaf. Let $T' = T - T_u$. Let D' be a $\gamma_{ev}(T')$-set. It is easy to see that $D' \cup \{uv\}$ is an EVDS of the tree T. Thus $\gamma_{ev}(T) \leq \gamma_{ev}(T') + 1$. Now let D be a $\gamma_t(T)$-set that contains no leaf. By Observation 1 we have $v, d \in D$. The vertex v has to be dominated, thus $u \in D$. Let us observe that $D \setminus \{u, v\}$ is a TDS of the tree T'. Therefore $\gamma_t(T') \leq \gamma_t(T) - 2$. We now get $\gamma_t(T') \leq \gamma_t(T) - 2 = \gamma_{ev}(T) - 1 \leq \gamma_{ev}(T') < \gamma_{ev}(T') + 1$.

Now assume that some child of d other than w, say k, is not a leaf. It suffices to consider the possibilities when T_k is isomorphic to T_w, or T_k is a path P_2 or P_3. Let $T' = T - T_w$. Let D' be a $\gamma_{ev}(T')$-set. It is easy to observe that $D' \cup \{dw, uv\}$ is an EVDS of the tree T. Thus $\gamma_{ev}(T) \leq \gamma_{ev}(T') + 2$. Now let D be a $\gamma_t(T)$-set that contains no leaf. Let us observe that some child of k has to be dominated by the vertex k. Therefore $k \in D$. By Observation 1 we have $v, w \in D$. The vertex v has to be dominated, thus $u \in D$. Let us observe that $D \setminus \{w, u, v\}$ is a TDS of the tree T'. Therefore $\gamma_t(T') \leq \gamma_t(T) - 3$. We now get $\gamma_t(T') \leq \gamma_t(T) - 3 = \gamma_{ev}(T) - 2 \leq \gamma_{ev}(T') < \gamma_{ev}(T') + 1$.

Now assume that $d_T(d) = 2$. If $d_T(e) = 1$, then we get $\gamma_t(T) = 4 > 3 = \gamma_{ev}(T) + 1$. Now assume that $d_T(e) \geq 2$. Let $T' = T - T_d$. Let D' be a $\gamma_{ev}(T')$-set. It is easy to observe that $D' \cup \{dw, uv\}$ is an EVDS of the tree T. Thus $\gamma_{ev}(T) \leq \gamma_{ev}(T') + 2$. Now let us observe that there exists a $\gamma_t(T)$-set that does not contain the vertices t, x and d. Let D be such a set. By Observation 1 we have $v, w \in D$. The vertex v has to be dominated, thus $u \in D$. Observe that $D \setminus \{w, u, v\}$ is a TDS of the tree T'. Therefore $\gamma_t(T') \leq \gamma_t(T) - 3$. We now get $\gamma_t(T') \leq \gamma_t(T) - 3 = \gamma_{ev}(T) - 2 \leq \gamma_{ev}(T') < \gamma_{ev}(T') + 1$. ■

As an immediate consequence of Lemmas 5 and 6, we have the following characterization of trees with total domination number equal to edge-vertex domination number plus one.

Theorem 7 *Let T be a tree. Then $\gamma_t(T) = \gamma_{ev}(T) + 1$ if and only if $T \in \mathcal{T}$.*

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