Minimal 2-dominating sets in trees

Marcin Krzywkowski*
e-mail: marcin.krzywkowski@gmail.com

Faculty of Electronics, Telecommunications and Informatics
Gdańsk University of Technology
Narutowicza 11/12, 80–233 Gdańsk, Poland

Abstract

We provide an algorithm for listing all minimal 2-dominating sets of a tree of order $n$ in time $O(1.3248^n)$. This implies that every tree has at most $1.3248^n$ minimal 2-dominating sets. We also show that this bound is tight.

Keywords: domination, 2-domination, minimal 2-dominating set, tree, counting, exact exponential algorithm, listing algorithm.

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1 Introduction

Let $G = (V, E)$ be a graph. The order of a graph is the number of its vertices. By the neighborhood of a vertex $v$ of $G$ we mean the set $N_G(v) = \{u \in V(G) : uv \in E(G)\}$. The degree of a vertex $v$, denoted by $d_G(v)$, is the cardinality of its neighborhood. By a leaf we mean a vertex of degree one, while a support vertex is a vertex adjacent to a leaf. The distance between two vertices of a graph is the number of edges in a shortest path connecting them. The eccentricity of a vertex is the greatest distance between it and any other vertex. The diameter of a graph $G$, denoted by $diam(G)$, is the maximum eccentricity

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among all vertices of $G$. By $P_n$ we denote a path on $n$ vertices. By a star we mean a connected graph in which exactly one vertex has degree greater than one.

A subset $D \subseteq V(G)$ is a dominating set of $G$ if every vertex of $V(G) \setminus D$ has a neighbor in $D$, while it is a 2-dominating set of $G$ if every vertex of $V(G) \setminus D$ has at least two neighbors in $D$. A dominating (2-dominating, respectively) set $D$ is minimal if no proper subset of $D$ is a dominating (2-dominating, respectively) set of $G$. A minimal 2-dominating set is abbreviated as m2ds. Note that 2-domination is a type of multiple domination in which each vertex, which is not in the dominating set, is dominated at least $k$ times for a fixed positive integer $k$.

Multiple domination was introduced by Fink and Jacobson [7], and further studied for example in [2, 10, 17]. For a comprehensive survey of domination in graphs, see [11, 12].

**Observation 1** Every leaf of a graph $G$ is in every 2-dominating set of $G$.

One of the typical questions in graph theory is how many subgraphs of a given property can a graph on $n$ vertices have. For example, the famous Moon and Moser theorem [16] says that every graph on $n$ vertices has at most $3^{n/3}$ maximal independent sets.

Combinatorial bounds are of interest not only on their own, but also because they are used for algorithm design as well. Lawler [15] used the Moon-Moser bound on the number of maximal independent sets to construct an $(1 + \sqrt{3})^n - n^{O(1)}$ time graph coloring algorithm, which was the fastest one known for twenty-five years. In 2003 Eppstein [6] reduced the running time of a graph coloring to $O(2.4151^n)$. In 2006 the running time was reduced [1, 14] to $O(2^n)$. For an overview of the field, see [9].

Fomin et al. [8] constructed an algorithm for listing all minimal dominating sets of a graph on $n$ vertices in time $O(1.7159^n)$. There were also given graphs ($n/6$ disjoint copies of the octahedron) having $15^{n/6} \approx 1.5704^n$ minimal dominating sets. This establishes a lower bound on the running time of an algorithm for listing all minimal dominating sets of a given graph.

The number of maximal independent sets in trees was investigated in [18]. Couturier et al. [5] considered minimal dominating sets in various classes of graphs. The authors of [13] investigated the enumeration of minimal dominating sets in graphs.

They also characterized the extremal trees. The authors of [4] investigated the number of minimal dominating sets in trees containing all leaves.

We provide an algorithm for listing all minimal 2-dominating sets of a tree of order $n$ in time $O(1.3248^n)$. This implies that every tree has at most $1.3248^n$ minimal 2-dominating sets. We also show that this bound is tight.

## 2 Listing algorithm

In this section we describe a recursive algorithm which lists all minimal 2-dominating sets of a given input tree $T$. The iterator of the solutions is denoted by $F(T)$.

### Algorithm

Notice that the diameter of a tree can be easily determined in a polynomial time.

Let $T$ be a tree. If $diam(T) = 0$, then $T = P_1 = v_1$. Let $F(T) = \{\{v_1\}\}$. If $diam(T) = 1$, then $T = P_2 = v_1v_2$. Let $F(T) = \{\{v_1, v_2\}\}$. If $diam(T) = 2$, then $T$ is a star. By $x$ we denote the support vertex of $T$. Let $F(T) = \{V(T) \setminus \{x\}\}$.

Now consider trees $T$ with $diam(T) \geq 3$. Thus the order $n$ of the tree $T$ is at least four.

If some support vertex of $T$, say $x$, is adjacent to at least three leaves (we denote one of them by $y$), then let $T' = T - y$ and

$$F(T) = \{D' \cup \{y\}: D' \in F(T')\}.$$  

Now consider trees $T$, in which every support vertex is adjacent to at most two leaves. The tree $T$ can easily be rooted at a vertex $r$ of maximum eccentricity $diam(T)$ in polynomial time. A leaf, say $t$, at maximum distance from $r$, can also be easily computed in polynomial time. Let $v$ denote the parent of $t$ and let $u$ denote the parent of $u$. By $T_x$ we denote the subtree induced by a vertex $x$ and its descendants in the rooted tree $T$.

If $d_T(v) = 3$, then by $a$ we denote the leaf adjacent to $v$ and different from $t$. Let $T' = T - T_v$ and $T'' = T - t - a$, and let $F(T)$ be as follows,

$$\{D' \cup \{t, a\}: D' \in F(T')\}$$

$$\cup \{D'' \cup \{t, a\}: D'' \in F(T'') \text{ and } D'' \setminus \{v\} \notin F(T')\}.$$
If \( d_T(v) = 2 \) and \( d_T(u) \geq 3 \), then let \( T' = T - T_v, T'' = T - T_u \), and
\[
\mathcal{F}(T) = \{ D' \cup \{ t \} : u \in D' \in \mathcal{F}(T') \} \cup \{ D'' \cup V(T_u) \setminus \{ u \} : D'' \in \mathcal{F}(T'') \}.
\]
If \( d_T(v) = d_T(u) = 2 \), then let \( T' = T - T_v, T'' = T - T_u \), and
\[
\mathcal{F}(T) = \{ D' \cup \{ t \} : D' \in \mathcal{F}(T') \} \cup \{ D'' \cup \{ v, t \} : w \in D'' \in \mathcal{F}(T'') \}.
\]

3 Bounding the number of minimal 2-dominating sets

Now we prove that the running time of the algorithm from the previous section is \( O(1.3248^n) \).

**Theorem 2** For every tree \( T \) of order \( n \), the algorithm from the previous section lists all minimal 2-dominating sets in time \( O(1.3248^n) \).

**Proof.** We prove that the running time of the algorithm is \( O(1.3248^n) \). Moreover, we prove that the number of minimal 2-dominating sets of \( T \) is at most \( \alpha^n \), where \( \alpha \approx 1.3248 \) is the positive solution of the equation \( x^3 - x - 1 = 0 \).

We proceed by induction on the number \( n \) of vertices of a tree \( T \). If \( \text{diam}(T) = 0 \), then \( T = P_1 = v_1 \). Obviously, \( \{ v_1 \} \) is the only \( m2ds \) of the path \( P_1 \). We have \( n = 1 \) and \( |\mathcal{F}(T)| = 1 \). Obviously, \( 1 < \alpha \). If \( \text{diam}(T) = 1 \), then \( T = P_2 = v_1v_2 \).

It is easy to see that \( \{ v_1, v_2 \} \) is the only \( m2ds \) of the path \( P_2 \). We have \( n = 2 \) and \( |\mathcal{F}(T)| = 1 \). Obviously, \( 1 < \alpha^2 \). If \( \text{diam}(T) = 2 \), then \( T \) is a star. By \( x \) we denote the support vertex of \( T \). It is easy to observe that \( V(T) \setminus \{ x \} \) is the only \( m2ds \) of the tree \( T \). We have \( n \geq 3 \) and \( |\mathcal{F}(T)| = 1 \). Obviously, \( 1 < \alpha^n \).

Now assume that \( \text{diam}(T) \geq 3 \). Thus the order \( n \) of the tree \( T \) is at least four. The results we obtain by the induction on the number \( n \). Assume that they are true for every tree \( T' \) of order \( n' < n \).

First assume that some support vertex of \( T \), say \( x \), is adjacent to at least three leaves. Let \( y \) be a leaf adjacent to \( x \). Let \( T' = T - y \). Let \( D' \) be a \( m2ds \) of the tree \( T' \). Obviously, \( D' \cup \{ y \} \) is an \( m2ds \) of \( T \). Thus all elements of \( \mathcal{F}(T) \) are minimal 2-dominating sets of the tree \( T \). Now let \( D \) be any \( m2ds \) of the tree \( T \). By Observation 1 we have \( y \in D \). Let us observe that \( D \setminus \{ y \} \) is an \( m2ds \) of the tree \( T' \) as the vertex \( x \) is still dominated at least twice. By the inductive
hypothesis we have $D \setminus \{y\} \in \mathcal{F}(T')$. Therefore $\mathcal{F}(T)$ contains all minimal 2-dominating sets of the tree $T$. Now we get $|\mathcal{F}(T)| = |\mathcal{F}(T')| \leq \alpha^{n-1} < \alpha^n$. Henceforth, we can assume that every support vertex of $T$ is adjacent to at most two leaves.

We now root $T$ at a vertex $r$ of maximum eccentricity $\text{diam}(T)$. Let $t$ be a leaf at maximum distance from $r$, $v$ be the parent of $t$, and $u$ be the parent of $v$ in the rooted tree. If $\text{diam}(T) \geq 4$, then let $w$ be the parent of $u$. By $T_x$ we denote the subtree induced by a vertex $x$ and its descendants in the rooted tree $T$.

Assume that $d_T(v) = 3$. By $a$ we denote the leaf adjacent to $v$ and different from $t$. Let $T' = T - T_v$ and $T'' = T - t - a$. Let us observe that all elements of $\mathcal{F}(T)$ are minimal 2-dominating sets of the tree $T$. Now let $D$ be any m2ds of the tree $T$. By Observation 1 we have $t, a \in D$. If $v \notin D$, then observe that $D \setminus \{t, a\}$ is an m2ds of the tree $T'$. By the inductive hypothesis we have $D \setminus \{t, a\} \in \mathcal{F}(T')$. Now assume that $v \in D$. Let us observe that $D \setminus \{t, a\}$ is an m2ds of the tree $T''$. By the inductive hypothesis we have $D \setminus \{t, a\} \in \mathcal{F}(T'')$. The set $D \setminus \{v, t, a\}$ is not an m2ds of the tree $T'$, otherwise $D \setminus \{v\}$ is a 2-dominating set of the tree $T$, a contradiction to the minimality of $D$. By the inductive hypothesis we have $D \setminus \{v, t, a\} \notin \mathcal{F}(T')$. Therefore $\mathcal{F}(T)$ contains all minimal 2-dominating sets of the tree $T$. Now we get $|\mathcal{F}(T)| = |\mathcal{F}(T')| + |D'' \setminus \{v\} \notin \mathcal{F}(T')| \leq |\mathcal{F}(T')| + |\mathcal{F}(T'')| \leq \alpha^{n-3} + \alpha^{n-2} = \alpha^{n-3}(\alpha + 1) = \alpha^{n-3} \cdot \alpha^3 = \alpha^n$.

Now assume that $d_T(v) = 2$. Assume that $d_T(u) \geq 3$. Let $T' = T - T_v$ and $T'' = T - T_u$. Let us observe that all elements of $\mathcal{F}(T)$ are minimal 2-dominating sets of the tree $T$. Now let $D$ be any m2ds of the tree $T$. By Observation 1 we have $t \in D$. If $v \notin D$, then $u \in D$ as the vertex $v$ has to be dominated twice. Observe that $D \setminus \{t\}$ is an m2ds of the tree $T'$. By the inductive hypothesis we have $D \setminus \{t\} \in \mathcal{F}(T')$. Now assume that $v \in D$. We have $u \notin D$, otherwise $D \setminus \{v\}$ is a 2-dominating set of the tree $T$, a contradiction to the minimality of $D$. Observe that $D \cap V(T'')$ is an m2ds of the tree $T''$. By the inductive hypothesis we have $D \cap V(T'') \in \mathcal{F}(T'')$. Therefore $\mathcal{F}(T)$ contains all minimal 2-dominating sets of the tree $T$. Now we get $|\mathcal{F}(T)| \leq |\mathcal{F}(T')| + |\mathcal{F}(T'')| \leq \alpha^{n-2} + \alpha^{n-3} = \alpha^{n-3}(\alpha + 1) = \alpha^{n-3} \cdot \alpha^3 = \alpha^n$.

Now assume that $d_T(u) = 2$. Let $T' = T - T_v$ and $T'' = T - T_u$. Let us observe that all elements of $\mathcal{F}(T)$ are minimal 2-dominating sets of the tree $T$. Now let $D$ be any m2ds of the tree $T$. By Observation 1 we have $t \in D$. If $v \notin D$, then observe that $D \setminus \{t\}$ is an m2ds of the tree $T'$. By the inductive hypothesis
we have \( D \setminus \{ t \} \in \mathcal{F}(T') \). Now assume that \( v \in D \). We have \( u \notin D \), otherwise \( D \setminus \{ v \} \) is a 2-dominating set of the tree \( T \), a contradiction to the minimality of \( D \). Moreover, we have \( w \in D \) as the vertex \( u \) has to be dominated twice. Observe that \( D \setminus \{ v, t \} \) is an m2ds of the tree \( T'' \). By the inductive hypothesis we have \( D \setminus \{ v, t \} \in \mathcal{F}(T'') \). Therefore \( \mathcal{F}(T) \) contains all minimal 2-dominating sets of the tree \( T \). Now we get \(|\mathcal{F}(T)| \leq |\mathcal{F}(T')| + |\mathcal{F}(T'')| \leq \alpha^{n-2} + \alpha^{n-3} = \alpha^{n-3}(\alpha + 1) = \alpha^{n-3} \cdot \alpha^3 = \alpha^n \).

It follows from the proof of the above theorem that any tree of order \( n \) has at most \( 1.3248n \) minimal 2-dominating sets.

**Corollary 3** Every tree of order \( n \) has at most \( \alpha^n \) minimal 2-dominating sets, where \( \alpha \approx 1.3248 \) is the positive solution of the equation \( x^3 - x - 1 = 0 \).

Now we show that the bound from the previous corollary is tight. Let \( a_n \) denote the number of minimal 2-dominating sets of the path \( P_n \). The next remark follows from the proof of Theorem 2.

**Remark 4** For every positive integer \( n \) we have

\[
a_n = \begin{cases} 
1 & \text{if } n \leq 3; \\
 a_{n-3} + a_{n-2} & \text{if } n \geq 4.
\end{cases}
\]

We have \( \lim_{n \to \infty} \sqrt[n]{a_n} = \alpha \), where \( \alpha \approx 1.3247 \) is the positive solution of the equation \( x^3 - x - 1 = 0 \). This implies that the bound from Corollary 3 is tight.

**References**


