# On trees with equal total domination and 2-outer-independent domination numbers

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**Abstract**

For a graph $G = (V, E)$, a subset $D \subseteq V(G)$ is a total dominating set if every vertex of $G$ has a neighbor in $D$. The total domination number of $G$ is the minimum cardinality of a total dominating set of $G$. A subset $D \subseteq V(G)$ is a 2-dominating set of $G$ if every vertex of $V(G) \setminus D$ has at least two neighbors in $D$, while it is a 2-outer-independent dominating set of $G$ if additionally the set $V(G) \setminus D$ is independent. The 2-outer-independent domination number of $G$ is the minimum cardinality of a 2-outer-independent dominating set of $G$. We characterize all trees with equal total domination and 2-outer-independent domination numbers.

**Keywords:** total domination, 2-outer-independent domination, 2-domination, tree.

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## 1 Introduction

Let $G = (V, E)$ be a graph. By the neighborhood of a vertex $v$ of $G$ we mean the set $N_G(v) = \{u \in V(G) \colon uv \in E(G)\}$. The degree of a vertex $v$, denoted by $d_G(v)$, is the cardinality of its neighborhood. By a leaf we mean a vertex of degree one, while a support vertex is a vertex adjacent to a leaf. We say that a support vertex is strong (weak, respectively) if it is adjacent to at least two leaves (exactly one leaf, respectively). We say that a subset of $V(G)$ is independent if there is no edge between any two vertices of this set. The path on $n$ vertices we denote by $P_n$. Let $T$ be a tree, and let $v$ be a vertex of $T$. We say that $v$ is adjacent to a tree $H$ if there is a neighbor of $v$, say $x$, such that the tree resulting from $T$ by removing the edge $vx$, and which contains the vertex $x$, is a tree $H$. We say that $v$ is adjacent to a path $P_n$ if there is a neighbor of $v$, say $x$, such that the subtree resulting from $T$ by removing the edge $vx$ and which contains the vertex $x$ as a leaf, is a path $P_n$. By a star we mean a connected graph in which exactly one vertex has degree greater than one.

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A subset $D \subseteq V(G)$ is a dominating set of $G$ if every vertex of $V(G) \setminus D$ has a neighbor in $D$, while it is a total dominating set, abbreviated TDS, of $G$ if every vertex of $G$ has a neighbor in $D$. The domination (total domination, respectively) number of $G$, denoted by $\gamma(G)$ ($\gamma_t(G)$, respectively), is the minimum cardinality of a dominating (total dominating, respectively) set of $G$. A total dominating set of $G$ of minimum cardinality is called a $\gamma_t(G)$-set. Total domination in graphs was introduced by Cockayne, Dawes, and Hedetniemi [6], and further studied for example in [1–3, 6–8, 10, 15, 19, 20]. For a comprehensive survey of domination in graphs, see [14].

A subset $D \subseteq V(G)$ is a 2-dominating set, abbreviated 2DS, of $G$ if every vertex of $V(G) \setminus D$ has at least two neighbors in $D$. The 2-domination number of $G$, denoted by $\gamma_2(G)$, is the minimum cardinality of a 2-dominating set of $G$. Note that 2-domination is a type of multiple domination in which each vertex, which is not in the dominating set, is dominated at least $k$ times for a fixed positive integer $k$. Multiple domination in graphs was introduced by Fink and Jacobson [9], and further studied for example in [4, 5, 9, 11, 12, 16, 18].

A subset $D \subseteq V(G)$ is a 2-outer-independent dominating set, abbreviated 2OIDS, of $G$ if every vertex of $V(G) \setminus D$ has at least two neighbors in $D$ and the set $V(G) \setminus D$ is independent. The 2-outer-independent domination number of $G$, denoted by $\gamma_2^{oi}(G)$, is the minimum cardinality of a 2-outer-independent dominating set of $G$. A 2-outer-independent dominating set of $G$ of minimum cardinality is called a $\gamma_2^{oi}(G)$-set. The study of 2-outer-independent domination in graphs was initiated in [17].

We characterize all trees with equal total domination and 2-outer-independent domination numbers.

## 2 Results

Since the one-vertex graph does not have a total dominating set, in this paper, by a tree we mean only a connected graph with no cycle, and which has at least two vertices.

We begin with the following three straightforward observations.

**Observation 1** *Every support vertex of a graph $G$ is in every TDS of $G$.*

**Observation 2** *For every connected graph $G$ of diameter at least three there exists a $\gamma_t(G)$-set that contains no leaf.*

**Observation 3** *Every leaf of a graph $G$ is in every 2OIDS of $G$.*

We now prove that the 2-outer-independent domination number of any tree is greater than or equal to its total domination number.

**Lemma 4** *For every tree $T$ we have $\gamma_2^{oi}(T) \geq \gamma_t(T)$.*

**Proof.** Since every 2OIDS of a tree $T$ is a 2DS of this tree, we have $\gamma_2^{oi}(T) \geq \gamma_2(T)$. In [13] it is proved that for every tree $T$ we have $\gamma_2(T) \geq \gamma_t(T)$. We now get $\gamma_2^{oi}(T) \geq \gamma_2(T) \geq \gamma_t(T)$. ■

We characterize all trees with equal total domination and 2-outer-independent domination numbers. For this purpose we introduce a family $\mathcal{T}$ of trees $T = T_k$ that can be obtained as follows. Let $T_1 \in \{P_2, P_3\}$. If $k$ is a positive integer, then $T_{k+1}$ can be obtained recursively from $T_k$ by one of the following operations.

- Operation $\mathcal{O}_1$: Attach a path $P_2$ by joining one of its vertices to a vertex of $T_k$ adjacent to a path $P_2$.
- Operation $\mathcal{O}_2$: Attach a path $P_3$ by joining one of its leaves to a vertex of $T_k$, which is not a leaf and is adjacent to a path $P_3$ or to a support vertex.
- Operation $\mathcal{O}_3$: Attach a path $P_5$ by joining one of its support vertices to a vertex of $T_k$, which is adjacent to a path $P_2$ or to a path $P_5$ through a support vertex.
- Operation $\mathcal{O}_4$: Attach a path $P_4$ by joining one of its leaves to a vertex of $T_k$, which is a leaf, or is adjacent to a path $P_2$ or $P_4$ or to a path $P_5$ through a support vertex.
- Operation $\mathcal{O}_5$: Let $x$ be a vertex of $T_k$ adjacent to a leaf, say $y$, and to a path $P_4$, say $abcd$. Let $a$ and $x$ be adjacent. Remove the leaf $y$. Then either attach a path $P_3$ by joining one of its leaves to the vertex $c$, or attach a path $P_3$ by joining one of its leaves to the vertex $b$ and a path $P_2$ by joining one of its vertices to the vertex $d$.
- Operation $\mathcal{O}_6$: Let $x$ be a support vertex of $T_k \neq P_5$ adjacent to a path $P_3$. Remove the path and a leaf adjacent to $x$, and attach a path $P_6$ by joining one of its support vertices to the vertex $x$.

We now prove that for every tree of the family $\mathcal{T}$, the total domination and the 2-outer-independent domination numbers are equal.

**Lemma 5** *If $T \in \mathcal{T}$, then $\gamma_t(T) = \gamma_2^{oi}(T)$.*

**Proof.** We use the induction on the number $k$ of operations performed to construct the tree $T$. If $T = P_2$, then obviously $\gamma_t(T) = 2 = \gamma_2^{oi}(T)$. If $T = P_3$, then also $\gamma_t(T) = 2 = \gamma_2^{oi}(T)$. Let $k$ be a positive integer. Assume that the result is true for every tree $T' = T_k$ of the family $\mathcal{T}$ constructed by $k-1$ operations. Let $T = T_{k+1}$ be a tree of the family $\mathcal{T}$ constructed by $k$ operations.

First assume that $T$ is obtained from $T'$ by operation $\mathcal{O}_1$. The vertex to which is attached $P_2$ we denote by $x$. Let $v_1v_2$ be the attached path. Let $v_1$ be joined to $x$. Let $yz$ be a path $P_2$ adjacent to $x$ and different from $v_1v_2$. Let $x$ and $y$ be adjacent. Let us observe that there exists a $\gamma_2^{oi}(T')$-set that contains the vertex $x$. Let $D'$ be such a set. It is easy to see that $D' \cup \{v_2\}$ is a 2OIDS of the tree $T$. Thus $\gamma_2^{oi}(T) \leq \gamma_2^{oi}(T') + 1$. Now let $D$ be a $\gamma_t(T)$-set that contains no leaf. By Observation 1 we have $v_1, y \in D$. Let us observe that $D \setminus \{v_1\}$ is a TDS of the tree $T'$ as the vertex $x$ has a neighbor in $D \setminus \{v_1\}$. Therefore $\gamma_t(T') \le \gamma_t(T) - 1$. We now get $\gamma_2^{oi}(T) \le \gamma_2^{oi}(T') + 1 = \gamma_t(T') + 1 \le \gamma_t(T)$. On the other hand, by Lemma 4 we have $\gamma_2^{oi}(T) \ge \gamma_t(T)$. This implies that $\gamma_2^{oi}(T) = \gamma_t(T)$.

Now assume that $T$ is obtained from $T'$ by operation $\mathcal{O}_2$. The vertex to which is attached $P_3$ we denote by $x$. Let $v_1v_2v_3$ be the attached path. Let $v_1$ be joined to $x$. Let $D'$ be a $\gamma_2^{oi}(T')$-set. It is easy to observe that $D' \cup \{v_1, v_3\}$ is a 2OIDS of the tree $T$. Thus $\gamma_2^{oi}(T) \le \gamma_2^{oi}(T') + 2$. First assume that $x$ is adjacent to a path $P_3$, say $abc$. Let $a$ and $x$ be adjacent. Let $D$ be a $\gamma_t(T)$-set that contains no leaf. By Observation 1 we have $v_2 \in D$. Each one of the vertices $v_2$ and $b$ has to be dominated, thus $v_1, a \in D$. Let us observe that $D \setminus \{v_1, v_2\}$ is a TDS of the tree $T'$. Now assume that $x$ is adjacent to a support vertex, say $y$. Let $D$ be a $\gamma_t(T)$-set that contains no leaf. By Observation 1 we have $v_2, y \in D$. The vertex $v_2$ has to be dominated, thus $v_1 \in D$. Let us observe that now also $D \setminus \{v_1, v_2\}$ is a TDS of the tree $T'$. We conclude that $\gamma_t(T') \le \gamma_t(T) - 2$. We now get $\gamma_2^{oi}(T) \le \gamma_2^{oi}(T') + 2 = \gamma_t(T') + 2 \le \gamma_t(T)$. This implies that $\gamma_2^{oi}(T) = \gamma_t(T)$.

Now assume that $T$ is obtained from $T'$ by operation $\mathcal{O}_3$. The vertex to which is attached $P_5$ we denote by $x$. Let $v_1v_2v_3v_4v_5$ be the attached path. Let $v_2$ be joined to $x$. Let $y$ be a support vertex adjacent to $x$ and different from $v_2$. Let us observe that there exists a $\gamma_2^{oi}(T')$-set that contains the vertex $x$. Let $D'$ be such a set. It is easy to observe that $D' \cup \{v_1, v_3, v_5\}$ is a 2OIDS of the tree $T$. Thus $\gamma_2^{oi}(T) \le \gamma_2^{oi}(T') + 3$. Now let $D$ be a $\gamma_t(T)$-set that contains no leaf. By Observation 1 we have $v_4, v_2, y \in D$. The vertex $v_4$ has to be dominated, thus $v_3 \in D$. Let us observe that $D \setminus \{v_2, v_3, v_4\}$ is a TDS of the tree $T'$. Therefore $\gamma_t(T') \le \gamma_t(T) - 3$. We now get $\gamma_2^{oi}(T) \le \gamma_2^{oi}(T') + 3 = \gamma_t(T') + 3 \le \gamma_t(T)$. This implies that $\gamma_2^{oi}(T) = \gamma_t(T)$.

Now assume that $T$ is obtained from $T'$ by operation $\mathcal{O}_4$. The vertex to which is attached $P_4$ we denote by $x$. Let $v_1v_2v_3v_4$ be the attached path. Let $v_1$ be joined to $x$. Let us observe that there exists a $\gamma_2^{oi}(T')$-set that contains the vertex $x$. Let $D'$ be such a set. It is easy to observe that $D' \cup \{v_2, v_4\}$ is a 2OIDS of the tree $T$. Thus $\gamma_2^{oi}(T) \le \gamma_2^{oi}(T') + 2$. Now let us observe that there exists a $\gamma_t(T)$-set that does not contain the vertices $v_4$ and $v_1$. Let $D$ be such a set. By Observation 1 we have $v_3 \in D$. The vertex $v_3$ has to be dominated, thus $v_2 \in D$. Observe that $D \setminus \{v_2, v_3\}$ is a TDS of the tree $T'$. Therefore $\gamma_t(T') \le \gamma_t(T) - 2$. We now get $\gamma_2^{oi}(T) \le \gamma_2^{oi}(T') + 2 = \gamma_t(T') + 2 \le \gamma_t(T)$. This implies that $\gamma_2^{oi}(T) = \gamma_t(T)$.

Now assume that $T$ is obtained from $T'$ by operation $\mathcal{O}_5$. The attached path $P_3$ we denote by $v_1v_2v_3$. Let $v_1$ be joined to $b$ or $c$. If we also attach a path $P_2$, then we denote it by $v_4v_5$. Let $v_4$ be joined to $d$. Let $D'$ be a $\gamma_2^{oi}(T')$-set that contains the vertices $b$ and $x$. By Observation 3 we have $d, y \in D'$. If we only attach a path $P_3$, then let us observe that $D' \setminus \{y\} \cup \{v_1, v_3\}$ is a 2OIDS of the tree $T$. If we also attach a path $P_2$, then let us observe that $D' \setminus \{y\} \cup \{v_1, v_3, v_5\}$ is a 2OIDS of the tree $T$. If we only attach a path $P_3$, then let us observe that there exists a $\gamma_t(T)$-set that does not contain the vertices $v_3$, $d$, $b$ and $a$. Let $D$ be such a set. By Observation 1 we have $v_2, c \in D$.

Each one of the vertices $v_2$ and $a$ has to be dominated, thus $v_1, x \in D$. Let us observe that $D \cup \{b\} \setminus \{v_1, v_2\}$ is a TDS of the tree $T'$. If we also attach a path $P_2$, then let us observe that there exists a $\gamma_t(T)$-set that does not contain the vertices $v_5$, $v_3$, $c$, $b$ and $a$. Let $D$ be such a set. By Observation 1 we have $v_2, v_4 \in D$. Each one of the vertices $v_2$, $v_4$, and $a$ has to be dominated, thus $v_1, d, x \in D$. Let us observe that $D \cup \{b, c\} \setminus \{v_1, v_2, d, v_4\}$ is a TDS of the tree $T'$. We now conclude that $\gamma_2^{oi}(T) + \gamma_t(T') \leq \gamma_2^{oi}(T') + \gamma_t(T)$. This implies that $\gamma_2^{oi}(T) = \gamma_t(T)$.

Now assume that $T$ is obtained from $T'$ by operation $\mathcal{O}_6$. The attached path we denote by $v_1v_2v_3v_4v_5v_6$. Let $v_2$ be joined to $x$. Let $y$ be a leaf adjacent to $x$, and which is being removed. Let $abc$ denote a path $P_3$ adjacent to $x$, and which is being removed. Let $a$ and $x$ be adjacent. Let $D'$ be a $\gamma_2^{oi}(T')$-set that contains the vertex $a$. By Observation 3 we have $c, y \in D'$. The set $D'$ is minimal, thus $b \notin D'$. If $x \in D'$, then it is easy to observe that $D' \setminus \{a, c, y\} \cup \{v_1, v_2, v_4, v_6\}$ is a 2OIDS of the tree $T$. Now assume that $x \notin D'$. Since $T_k \neq P_5$, the vertex $x$ has at least three neighbors in the tree $T'$. Let $z$ be a neighbor of $x$ other than $a$ and $y$. We have $z \in D'$ as the set $V(T') \setminus D'$ is independent. Let us observe that now also $D' \setminus \{a, c, y\} \cup \{v_1, v_2, v_4, v_6\}$ is a 2OIDS of the tree $T$. Thus $\gamma_2^{oi}(T) \leq \gamma_2^{oi}(T') + 1$. Now let $D$ be a $\gamma_t(T)$-set that does not contain the vertices $v_6$, $v_3$ and $v_1$. Let $D$ be such a set. By Observation 3 we have $v_5, v_2 \in D$. Each one of the vertices $v_5$ and $v_2$ has to be dominated, thus $v_4, x \in D$. Let us observe that $D \cup \{a, b\} \setminus \{v_2, v_4, v_5\}$ is a TDS of the tree $T'$. Therefore $\gamma_t(T') \leq \gamma_t(T) - 1$. We now get $\gamma_2^{oi}(T) \leq \gamma_2^{oi}(T') + 1 = \gamma_t(T') + 1 \leq \gamma_t(T)$. This implies that $\gamma_2^{oi}(T) = \gamma_t(T)$. ∎

We now prove that if the total domination and the 2-outer-independent domination numbers of a tree are equal, then the tree belongs to the family $\mathcal{T}$.

**Lemma 6** *Let $T$ be a tree. If $\gamma_t(T) = \gamma_2^{oi}(T)$, then $T \in \mathcal{T}$.*

**Proof.** If $\text{diam}(T) = 1$, then $T = P_2 \in \mathcal{T}$. Now assume that $\text{diam}(T) = 2$. Thus $T$ is a star. If $T = P_3$, then $T \in \mathcal{T}$. Now assume that $T$ is a star different from $P_3$. We have $\gamma_t(T) = 2 < n - 1 = \gamma_2^{oi}(T)$.

Now assume that $\text{diam}(T) \geq 3$. Thus the order $n$ of the tree $T$ is at least four. We obtain the result by the induction on the number $n$. Assume that the lemma is true for every tree $T'$ of order $n' < n$.

First assume that some support vertex of $T$, say $x$, is strong. Let $y$ and $z$ be leaves adjacent to $x$. Let $T' = T - y$. Let $D'$ be a $\gamma_t(T')$-set. By Observation 1 we have $x \in D'$. It is easy to see that $D'$ is a TDS of the tree $T$. Thus $\gamma_t(T) \leq \gamma_t(T')$. Now let $D$ be a $\gamma_2^{oi}(T)$-set. By Observation 3 we have $y, z \in D$. If $x \in D$, then it is easy to observe that $D \setminus \{y\}$ is a 2OIDS of the tree $T'$. Now assume that $x \notin D$. Let $k$ be a neighbor of $x$ other than $y$ and $z$. The set $V(T) \setminus D$ is independent, thus $k \in D$. Let us observe that now also $D \setminus \{y\}$ is a 2OIDS of the tree $T'$ as the vertex $x$ has at least two neighbors in $D \setminus \{y\}$. Therefore $\gamma_2^{oi}(T') \leq \gamma_2^{oi}(T) - 1$. We now get $\gamma_t(T') \geq \gamma_t(T) = \gamma_2^{oi}(T) \geq \gamma_2^{oi}(T') + 1 > \gamma_2^{oi}(T')$. This is a contradiction as by Lemma 4 we have $\gamma_t(T') \leq \gamma_2^{oi}(T')$. Therefore every support vertex of $T$ is weak.

We now root $T$ at a vertex $r$ of maximum eccentricity $\mathrm{diam}(T)$. Let $t$ be a leaf at maximum distance from $r$, $v$ be the parent of $t$, and $u$ be the parent of $v$ in the rooted tree. If $\mathrm{diam}(T) \geq 4$, then let $w$ be the parent of $u$. If $\mathrm{diam}(T) \geq 5$, then let $d$ be the parent of $w$. If $\mathrm{diam}(T) \geq 6$, then let $e$ be the parent of $d$. If $\mathrm{diam}(T) \geq 7$, then let $f$ be the parent of $e$. By $T_x$ we denote the subtree induced by a vertex $x$ and its descendants in the rooted tree $T$.

Assume that $d_T(u) \geq 3$. Let $x$ be a child of $u$ other than $v$. First assume that $x$ is a leaf. Let $T' = T - x$. Let $D'$ be a $\gamma_t(T')$-set that contains no leaf. The vertex $v$ has to be dominated, thus $u \in D'$. It is easy to see that $D'$ is a TDS of the tree $T$. Thus $\gamma_t(T) \leq \gamma_t(T')$. Now let us observe that there exists a $\gamma_2^{oi}(T)$-set that contains the vertex $u$. Let $D$ be such a set. By Observation 3 we have $x \in D$. It is easy to observe that $D \setminus \{x\}$ is a 2OIDS of the tree $T'$. Therefore $\gamma_2^{oi}(T') \leq \gamma_2^{oi}(T) - 1$. We now get $\gamma_t(T') \geq \gamma_t(T) = \gamma_2^{oi}(T) \geq \gamma_2^{oi}(T') + 1 > \gamma_2^{oi}(T')$, a contradiction.

Thus $x$ is a support vertex of degree two. Let $T' = T - T_v$. Let $D'$ be a $\gamma_t(T')$-set that contains no leaf. The vertex $x$ has to be dominated, thus $u \in D'$. It is easy to see that $D' \cup \{v\}$ is a TDS of the tree $T$. Thus $\gamma_t(T) \leq \gamma_t(T') + 1$. Now let us observe that there exists a $\gamma_2^{oi}(T)$-set that does not contain the vertex $v$. Let $D$ be such a set. By Observation 3 we have $t \in D$. Observe that $D \setminus \{t\}$ is a 2OIDS of the tree $T'$. Therefore $\gamma_2^{oi}(T') \leq \gamma_2^{oi}(T) - 1$. We now get $\gamma_t(T') \geq \gamma_t(T) - 1 = \gamma_2^{oi}(T) - 1 \geq \gamma_2^{oi}(T')$. On the other hand, by Lemma 4 we have $\gamma_t(T') \leq \gamma_2^{oi}(T')$. This implies that $\gamma_t(T') = \gamma_2^{oi}(T')$. By the inductive hypothesis we have $T' \in \mathcal{T}$. The tree $T$ can be obtained from $T'$ by operation $\mathcal{O}_1$. Thus $T \in \mathcal{T}$.

Now assume that $d_T(u) = 2$. First assume that there is a child of $w$ other than $u$, say $k$, such that the distance of $w$ to the most distant vertex of $T_k$ is three or two. It suffices to consider only the possibilities when $T_k$ is a path $P_3$ or $P_2$. Let $T' = T - T_u$. Let $D'$ be a $\gamma_t(T')$-set. It is easy to observe that $D' \cup \{u, v\}$ is a TDS of the tree $T$. Thus $\gamma_t(T) \leq \gamma_t(T') + 2$. Now let $D$ be a $\gamma_2^{oi}(T)$-set that contains the vertex $u$. By Observation 3 we have $t \in D$. The set $D$ is minimal, thus $v \notin D$. If $w \in D$, then it is easy to observe that $D \setminus \{u, t\}$ is a 2OIDS of the tree $T'$. Now assume that $w \notin D$. The set $V(T) \setminus D$ is independent, thus $k, d \in D$. Let us observe that now also $D \setminus \{u, t\}$ is a 2OIDS of the tree $T'$ as the vertex $w$ has at least two neighbors in $D \setminus \{u, t\}$. Therefore $\gamma_2^{oi}(T') \leq \gamma_2^{oi}(T) - 2$. We now get $\gamma_t(T') \geq \gamma_t(T) - 2 = \gamma_2^{oi}(T) - 2 \geq \gamma_2^{oi}(T')$. This implies that $\gamma_t(T') = \gamma_2^{oi}(T')$. By the inductive hypothesis we have $T' \in \mathcal{T}$. The tree $T$ can be obtained from $T'$ by operation $\mathcal{O}_2$. Thus $T \in \mathcal{T}$.

Now assume that some child of $w$, say $x$, is a leaf. We can assume that $d_T(w) = 3$. First assume that there is a child of $d$ other than $w$, say $k$, such that the distance of $d$ to the most distant vertex of $T_k$ is four or two. It suffices to consider the possibilities when $T_k$ is isomorphic to $T_w$, or $T_k$ is a path $P_4$ or $P_2$. First assume that $T_k$ is isomorphic to $T_w$, or $T_k$ is a path $P_2$. Let $T' = T - T_w$. Let $D'$ be a $\gamma_t(T')$-set. It is easy to observe that $D' \cup \{w, u, v\}$ is a TDS of the tree $T$. Thus $\gamma_t(T) \leq \gamma_t(T') + 3$. Now let $D$ be a $\gamma_2^{oi}(T)$-set that does not contain the vertices $v$ and $w$. By Observation 3 we have $t, x \in D$.

The vertex $u$ has no neighbor in $D$, thus $u \in D$. Observe that $D \setminus \{u, t, x\}$ is a 2OIDS of the tree $T'$. Therefore $\gamma_2^{oi}(T') \leq \gamma_2^{oi}(T) - 3$. We now get $\gamma_t(T') \geq \gamma_t(T) - 3 = \gamma_2^{oi}(T) - 3 \geq \gamma_2^{oi}(T')$. This implies that $\gamma_t(T') = \gamma_2^{oi}(T')$. By the inductive hypothesis we have $T' \in \mathcal{T}$. The tree $T$ can be obtained from $T'$ by operation $\mathcal{O}_3$. Thus $T \in \mathcal{T}$.

Now assume that $T_k$ is a path $P_4$, say $klmp$. Let $T' = T - T_k$. Let $D'$ be a $\gamma_t(T')$-set. It is easy to observe that $D' \cup \{l, m\}$ is a TDS of the tree $T$. Thus $\gamma_t(T) \leq \gamma_t(T') + 2$. Now let $D$ be a $\gamma_2^{oi}(T)$-set that does not contain the vertices $m$ and $k$. By Observation 3 we have $p \in D$. The vertex $l$ has no neighbor in $D$, thus $l \in D$. Observe that $D \setminus \{l, p\}$ is a 2OIDS of the tree $T'$. Therefore $\gamma_2^{oi}(T') \leq \gamma_2^{oi}(T) - 2$. We now get $\gamma_t(T') \geq \gamma_t(T) - 2 = \gamma_2^{oi}(T) - 2 \geq \gamma_2^{oi}(T')$. This implies that $\gamma_t(T') = \gamma_2^{oi}(T')$. By the inductive hypothesis we have $T' \in \mathcal{T}$. The tree $T$ can be obtained from $T'$ by operation $\mathcal{O}_4$. Thus $T \in \mathcal{T}$.

Now assume that there is a child of $d$, say $k$, such that the distance of $d$ to the most distant vertex of $T_k$ is three. It suffices to consider only the possibility when $T_k$ is a path $P_3$, say $klm$. Let $T' = T - T_k$. Similarly as earlier we conclude that $\gamma_t(T) \leq \gamma_t(T') + 2$. Now let $D$ be a $\gamma_2^{oi}(T)$-set that contains the vertices $u$ and $d$. By Observation 3 we have $m \in D$. Without loss of generality we assume that $k \in D$ and $l \notin D$. It is easy to observe that $D \setminus \{k, m\}$ is a 2OIDS of the tree $T'$. Therefore $\gamma_2^{oi}(T') \leq \gamma_2^{oi}(T) - 2$. We now get $\gamma_t(T') \geq \gamma_t(T) - 2 = \gamma_2^{oi}(T) - 2 \geq \gamma_2^{oi}(T')$. This implies that $\gamma_t(T') = \gamma_2^{oi}(T')$. By the inductive hypothesis we have $T' \in \mathcal{T}$. The tree $T$ can be obtained from $T'$ by operation $\mathcal{O}_2$. Thus $T \in \mathcal{T}$.

Now assume that some child of $d$, say $k$, is a leaf. Let $T' = T - T_u - k$. Let $D'$ be a $\gamma_t(T')$-set that contains no leaf. By Observation 1 we have $w \in D'$. The vertex $w$ has to be dominated, thus $d \in D'$. It is easy to observe that $D' \cup \{u, v\}$ is a TDS of the tree $T$. Thus $\gamma_t(T) \leq \gamma_t(T') + 2$. Now let $D$ be a $\gamma_2^{oi}(T)$-set that contains the vertices $u$ and $d$. By Observation 3 we have $t, x, k \in D$. The set $D$ is minimal, thus $v \notin D$. Let us observe that $D \setminus \{u, t, k\}$ is a 2OIDS of the tree $T'$. Therefore $\gamma_2^{oi}(T') \leq \gamma_2^{oi}(T) - 3$. We now get $\gamma_t(T') \geq \gamma_t(T) - 2 = \gamma_2^{oi}(T) - 2 \geq \gamma_2^{oi}(T') + 1 > \gamma_2^{oi}(T')$, a contradiction.

Now assume that $d_T(d) = 2$. Assume that $d_T(e) \geq 3$. Let $T' = T - T_d$. Let $D'$ be a $\gamma_t(T')$-set. It is easy to observe that $D' \cup \{w, u, v\}$ is a TDS of the tree $T$. Thus $\gamma_t(T) \leq \gamma_t(T') + 3$. Now let $D$ be a $\gamma_2^{oi}(T)$-set that contains the vertices $u$ and $d$. By Observation 3 we have $t, x \in D$. The set $D$ is minimal, thus $v, w \notin D$. If $e \in D$, then it is easy to observe that $D \setminus \{d, u, t, x\}$ is a 2OIDS of the tree $T'$. Now assume that $e \notin D$. Let $k$ be a neighbor of $e$ other than $d$ and $f$. The set $V(T) \setminus D$ is independent, thus $f, k \in D$. Let us observe that now also $D \setminus \{d, u, t, x\}$ is a 2OIDS of the tree $T'$ as the vertex $e$ has at least two neighbors in $D \setminus \{d, u, t, x\}$. Therefore $\gamma_2^{oi}(T') \leq \gamma_2^{oi}(T) - 4$. We now get $\gamma_t(T') \geq \gamma_t(T) - 3 = \gamma_2^{oi}(T) - 3 \geq \gamma_2^{oi}(T') + 1 > \gamma_2^{oi}(T')$, a contradiction.

If $d_T(e) = 1$, then it is easy to verify that $\gamma_t(T) = 4 < 5 = \gamma_2^{oi}(T)$, a contradiction. Now assume that $d_T(e) = 2$. Let $T'$ be a tree obtained from $T - T_u$ by attaching a vertex, say $y$, by joining it to the vertex $f$. Let $D'$ be a $\gamma_t(T')$-set that contains no leaf. By Observation 1 we have $w, f \in D'$. The vertex $w$ has to be dominated, thus $d \in D'$. Let us observe that $D' \setminus \{d\} \cup \{u, v\}$

is a TDS of the tree $T$. Thus $\gamma_t(T) \leq \gamma_t(T') + 1$. Now let $D$ be a $\gamma_2^{oi}(T)$-set that contains the vertices $u$ and $d$. By Observation 3 we have $t, x \in D$. The set $D$ is minimal, thus $v \notin D$. Let us observe that $D \cup \{y\} \setminus \{u, t\}$ is a 2OIDS of the tree $T'$. Therefore $\gamma_2^{oi}(T') \leq \gamma_2^{oi}(T) - 1$. We now get $\gamma_t(T') \geq \gamma_t(T) - 1 = \gamma_2^{oi}(T) - 1 \geq \gamma_2^{oi}(T')$. This implies that $\gamma_t(T') = \gamma_2^{oi}(T')$. By the inductive hypothesis we have $T' \in \mathcal{T}$. The tree $T$ can be obtained from $T'$ by operation $\mathcal{O}_5$. Thus $T \in \mathcal{T}$.

If $d_T(w) = 1$, then $T = P_4$. We have $\gamma_t(T) = 2 < 3 = \gamma_2^{oi}(T)$. Now assume that $d_T(w) = 2$. If $d_T(d) = 1$, then $T = P_5$. Let $T' = P_3 \in \mathcal{T}$. The tree $T$ can be obtained from $T'$ by operation $\mathcal{O}_1$. Thus $T \in \mathcal{T}$. Now assume that $d_T(d) = 2$, or $d_T(d) \geq 3$ and there is a child of $d$ other than $w$, say $k$, such that the distance of $d$ to the most distant vertex of $T_k$ is four or two (then it suffices to consider only the possibilities when $T_k$ is a path $P_4$ or $P_2$). Let $T' = T - T_w$. Similarly as earlier we conclude that $\gamma_t(T) \leq \gamma_t(T') + 2$ and $\gamma_2^{oi}(T') \leq \gamma_2^{oi}(T) - 2$. We now get $\gamma_t(T') \geq \gamma_t(T) - 2 = \gamma_2^{oi}(T) - 2 \geq \gamma_2^{oi}(T')$. This implies that $\gamma_t(T') = \gamma_2^{oi}(T')$. By the inductive hypothesis we have $T' \in \mathcal{T}$. The tree $T$ can be obtained from $T'$ by operation $\mathcal{O}_4$. Thus $T \in \mathcal{T}$.

Now assume that some child of $d$, say $x$, is a leaf. Assume that there is also another child of $d$, say $k$, such that the distance of $d$ to the most distant vertex of $T_k$ is three. It suffices to consider only the possibility when $T_k$ is a path $P_3$, say $klm$. Let $T' = T - x - m$. Let $D'$ be a $\gamma_t(T')$-set that contains no leaf. By Observation 1 we have $k \in D'$. The vertex $k$ has to be dominated, thus $d \in D'$. It is easy to observe that $D' \cup \{l\}$ is a TDS of the tree $T$. Thus $\gamma_t(T) \leq \gamma_t(T') + 1$. Now let $D$ be a $\gamma_2^{oi}(T)$-set that contains the vertices $u$ and $d$. By Observation 3 we have $x, m \in D$. Without loss of generality we assume that $l \in D$ and $k \notin D$. Let us observe that $D \setminus \{x, m\}$ is a 2OIDS of the tree $T'$. Therefore $\gamma_2^{oi}(T') \leq \gamma_2^{oi}(T) - 2$. We now get $\gamma_t(T') \geq \gamma_t(T) - 1 = \gamma_2^{oi}(T) - 1 \geq \gamma_2^{oi}(T') + 1 > \gamma_2^{oi}(T')$, a contradiction.

Now assume that $d_T(d) = 3$. Let $T'$ be a tree obtained from $T - T_v - x$ by attaching a vertex, say $y$, by joining it to the vertex $e$. Let $D'$ be a $\gamma_t(T')$-set that contains no leaf. By Observation 1 we have $w, e \in D'$. The vertex $w$ has to be dominated, thus $d \in D'$. Let us observe that $D' \setminus \{w\} \cup \{u, v\}$ is a TDS of the tree $T$. Thus $\gamma_t(T) \leq \gamma_t(T') + 1$. Now let $D$ be a $\gamma_2^{oi}(T)$-set that contains the vertices $u$ and $d$. By Observation 3 we have $t, x \in D$. The set $D$ is minimal, thus $v \notin D$. Let us observe that $D \cup \{y\} \setminus \{t, x\}$ is a 2OIDS of the tree $T'$. Therefore $\gamma_2^{oi}(T') \leq \gamma_2^{oi}(T) - 1$. We now get $\gamma_t(T') \geq \gamma_t(T) - 1 = \gamma_2^{oi}(T) - 1 \geq \gamma_2^{oi}(T')$. This implies that $\gamma_t(T') = \gamma_2^{oi}(T')$. By the inductive hypothesis we have $T' \in \mathcal{T}$. The tree $T$ can be obtained from $T'$ by operation $\mathcal{O}_6$. Thus $T \in \mathcal{T}$.

Now assume that for every child of $d$ other than $w$, say $k$, the distance of $d$ to the most distant vertex of $T_k$ is three, and consequently, $T_k$ is a path $P_3$. Due to the earlier analysis of the children of the vertex $w$, we may assume that $d_T(d) = 3$. Assume that $d_T(e) \geq 3$. Let $T' = T - T_d$. Let $D'$ be a $\gamma_t(T')$-set. It is easy to observe that $D' \cup \{u, v, k, l\}$ is a TDS of the tree $T$. Thus $\gamma_t(T) \leq \gamma_t(T') + 4$. Now let $D$ be a $\gamma_2^{oi}(T)$-set that contains the vertices $u$ and $d$. By Observation 3 we have $t, m \in D$. The set $D$ is minimal, thus $v, w \notin D$. Without loss of generality we assume that $k \in D$ and $l \notin D$.

If $e \in D$, then it is easy to observe that $D \setminus \{d, u, t, k, m\}$ is a 2OIDS of the tree $T'$. Now assume that $e \notin D$. Let $x$ be a neighbor of $e$ other than $d$ and $f$. The set $V(T) \setminus D$ is independent, thus $f, x \in D$. Let us observe that now also $D \setminus \{d, u, t, k, m\}$ is a 2OIDS of the tree $T'$. Therefore $\gamma_2^{oi}(T') \leq \gamma_2^{oi}(T) - 5$. We now get $\gamma_t(T') \geq \gamma_t(T) - 4 = \gamma_2^{oi}(T) - 4 \geq \gamma_2^{oi}(T') + 1 > \gamma_2^{oi}(T')$, a contradiction.

Now assume that $d_T(e) = 2$. Let $T'$ be a tree obtained from $T - T_v - T_k$ by attaching a vertex, say $y$, by joining it to the vertex $f$. Let $D'$ be a $\gamma_t(T')$-set that contains no leaf. By Observation 1 we have $w, f \in D'$. The vertex $w$ has to be dominated, thus $d \in D'$. Let us observe that $D' \setminus \{d, w\} \cup \{u, v, k, l\}$ is a TDS of the tree $T$. Thus $\gamma_t(T) \leq \gamma_t(T') + 2$. Now let $D$ be a $\gamma_2^{oi}(T)$-set that contains the vertices $u$ and $d$. By Observation 3 we have $t, m \in D$. The set $D$ is minimal, thus $v \notin D$. Without loss of generality we assume that $k \in D$ and $l \notin D$. Let us observe that $D \cup \{y\} \setminus \{t, k, m\}$ is a 2OIDS of the tree $T'$. Therefore $\gamma_2^{oi}(T') \leq \gamma_2^{oi}(T) - 2$. We now get $\gamma_t(T') \geq \gamma_t(T) - 2 = \gamma_2^{oi}(T) - 2 \geq \gamma_2^{oi}(T')$. This implies that $\gamma_t(T') = \gamma_2^{oi}(T')$. By the inductive hypothesis we have $T' \in \mathcal{T}$. The tree $T$ can be obtained from $T'$ by operation $\mathcal{O}_5$. Thus $T \in \mathcal{T}$. ■

As an immediate consequence of Lemmas 5 and 6, we have the following characterization of the trees with total domination number equal to 2-outer-independent domination number.

**Theorem 7** *Let $T$ be a tree. Then $\gamma_t(T) = \gamma_2^{oi}(T)$ if and only if $T \in \mathcal{T}$.*

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