

On the Roman reinforcement in graphs

Nader Jafari Rad*

n.jafarirad@gmail.com

Marcin Krzywkowski^{†‡}

marcin.krzywkowski@gmail.com

Abstract

A Roman dominating function (RDF) on a graph G is a function $f: V(G) \rightarrow \{0, 1, 2\}$ satisfying the condition that every vertex v for which $f(v) = 0$, is adjacent to at least one vertex u for which $f(u) = 2$. The weight of a Roman dominating function f is the value $f(V(G)) = \sum_{v \in V(G)} f(v)$. The Roman domination number of G , denoted by $\gamma_R(G)$, is the minimum weight of an RDF on G . The Roman reinforcement number $r_R(G)$ of a graph G is the minimum number of edges that have to be added to G in order to decrease the Roman domination number. We first show that the Roman reinforcement problem is NP-complete even when restricted to bipartite graphs. Then we prove some sharp bounds on $r_R(G)$. Next we characterize trees with Roman reinforcement number greater than one. We also characterize graphs with Roman reinforcement number equal to the maximum degree.

Keywords: Roman domination, reinforcement, Roman reinforcement, complexity.

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1 Introduction

Let $G = (V, E)$ be a graph. The open neighborhood of a vertex v of G is the set $N_G(v) = \{u \in V(G) : uv \in E(G)\}$. The closed neighborhood of v is the set $N_G[v] = N_G(v) \cup \{v\}$. The degree of a vertex v , that is, the cardinality of its open neighborhood, is denoted by $d_G(v)$. By a leaf we mean a vertex of degree one, while a support vertex is a vertex adjacent to a leaf. We say that a support

*Department of Mathematics, Shahrood University of Technology, Iran

[†]Research fellow at the Department of Mathematics, University of Johannesburg, South Africa.

[‡]Faculty of Electronics, Telecommunications and Informatics, Gdansk University of Technology, Poland. Research partially supported by the Polish National Science Centre grant 2011/02/A/ST6/00201.

vertex is strong (weak, respectively) if it is adjacent to at least two leaves (exactly one leaf, respectively). Let $\Delta(G)$ mean the maximum degree among all vertices of G . The distance between two vertices of a graph is the number of edges in a shortest path connecting them. The eccentricity of a vertex is the greatest distance between it and any other vertex. The diameter of a graph G , denoted by $\text{diam}(G)$, is the maximum eccentricity among all vertices of G . The complete graph on n vertices we denote by K_n . The path (cycle, respectively) on n vertices we denote by P_n (C_n , respectively). Let T be a tree, and let v be a vertex of T . We say that v is adjacent to a path P_n if there is a neighbor of v , say x , such that the subtree resulting from T by removing the edge vx and which contains the vertex x as a leaf, is a path P_n . By a star we mean a connected graph in which exactly one vertex has degree greater than one. Double star is a graph obtained from a star by joining a positive number of vertices to one of the leaves.

A subset $D \subseteq V(G)$ is a dominating set of G if every vertex of $V(G) \setminus D$ has a neighbor in D . The domination number of G , denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G . An efficient dominating set of a graph G is a dominating set D of G such that the closed neighborhoods $N_G[v]$, where $v \in D$, form a partition of $V(G)$. For a comprehensive survey of domination in graphs, see [9].

A function $f: V(G) \rightarrow \{0, 1, 2\}$ is a Roman dominating function (RDF) on G if every vertex v for which $f(v) = 0$, is adjacent to at least one vertex u for which $f(u) = 2$. The weight of an RDF is the value $\omega(f) = \sum_{v \in V(G)} f(v)$. The Roman domination number of a graph G , denoted by $\gamma_R(G)$, is the minimum weight of an RDF on G . A function $f = (V_0, V_1, V_2)$ is called a $\gamma_R(G)$ -function if it is an RDF on G and $\omega(f) = \gamma_R(G)$. A Roman dominating function $f: V(G) \rightarrow \{0, 1, 2\}$ can be represented by the ordered partition (V_0, V_1, V_2) (or (V_0^f, V_1^f, V_2^f) to refer to f) of $V(G)$, where $V_i = \{v \in V(G): f(v) = i\}$. In this representation, the weight of the function is $\omega(f) = |V_1| + 2|V_2|$. The concept of Roman domination in graphs was introduced by Stewart [16], and further studied for example in [4, 7, 13, 15].

Kok and Mynhardt [14] introduced the reinforcement number $r(G)$ of a graph G as the minimum number of edges that have to be added to G in order that the resulting graph G' satisfies $\gamma(G') < \gamma(G)$. The concept of reinforcement number was further considered for several variants of domination, including independent domination, total domination and total restrained domination, see for example [1, 3, 5, 6, 11].

Jafari Rad and Sheikholeslami [12] studied the concept of Roman reinforcement in graphs. The Roman reinforcement number $r_R(G)$ of a graph G is the minimum number of edges that have to be added to G in order to decrease the Roman domination number. It is obvious that if $\gamma_R(G) \leq 2$, then addition of edges does not reduce the Roman domination number. In such case it is defined that $r_R(G) = 0$.

We first show that the Roman reinforcement problem is NP-complete even when restricted to bipartite graphs. Then we prove some sharp bounds on $r_R(G)$. Next we characterize trees with Roman reinforcement number greater than one. We also characterize graphs with Roman reinforcement number equal to the maximum degree.

2 Known results

Chambers et al. [2] proved the following upper bound on the Roman domination number of a graph.

Proposition 1 ([2]) *If G is a connected graph of order n , then $\gamma_R(G) \leq n - \Delta(G) + 1$.*

The authors of [12] obtained the following results.

Theorem 2 ([12]) *Let G be a connected graph of order $n \geq 3$. Then $r_R(G) = 1$ if and only if there is a $\gamma_R(G)$ -function $f = (V_0, V_1, V_2)$ such that $V_1 \neq \emptyset$.*

Theorem 3 ([12]) *If G is a graph without isolated vertices, then $r_R(G) \leq \Delta(G)$.*

Theorem 4 ([12]) *For any graph G of order n we have $r_R(G) \leq \lceil 2n/\gamma_R(G) \rceil - 1$.*

3 Complexity

In this section we prove that the Roman reinforcement decision problem is NP-complete, even when restricted to bipartite graphs. We shall prove the NP-completeness by reducing the following 3-SAT problem, which is known to be NP-complete [8]. The problem 3-SAT is the problem of determining whether there exists an interpretation that satisfies a given Boolean formula. The formula in 3-SAT is given in conjunctive normal form, where each clause contains three literals.

ROMAN REINFORCEMENT PROBLEM

INSTANCE: A graph $G = (V, E)$ and a positive integer k .

QUESTION: Is $r_R(G) \leq k$?

Theorem 5 *The Roman reinforcement problem is NP-complete for bipartite graphs.*

Proof. It is clear that the Roman reinforcement problem belongs to NP. We show the NP-hardness by transforming the 3-SAT problem to the Roman reinforcement problem in polynomial time. Let $U = \{u_1, u_2, \dots, u_n\}$ and $\mathcal{C} = \{C_1, C_2, \dots, C_m\}$

be an arbitrary instance of 3-SAT. We construct a bipartite graph G and an integer k such that \mathcal{C} is satisfiable if and only if $r_R(G) \leq k$. To each variable $u_i \in U$, we associate a graph H_i with $V(H_i) = \{u_i, \bar{u}_i, a_i, b_i, c_i, d_i, e_i, f_i\}$ and $E(H_i) = \{u_i d_i, u_i e_i, \bar{u}_i e_i, \bar{u}_i b_i, c_i e_i, c_i b_i, c_i d_i, a_i d_i, a_i b_i, d_i f_i, b_i f_i\}$, see Figure 1.

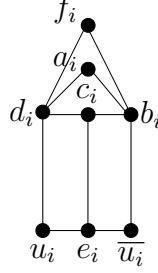


Figure 1: A graph H_i

With each clause $C_j = \{x_j, y_j, z_j\} \in \mathcal{C}$, associate a single vertex c_j and add the edge-set $E_j = \{c_j x_j, c_j y_j, c_j z_j\}$. Finally, add a path $P_2 = ss_1$ and join s to each vertex c_j with $1 \leq j \leq m$, and let $k = 1$. Let $f = (V_0^f, V_1^f, V_2^f)$ be a $\gamma_R(G)$ -function. Clearly, $\sum_{v \in V(H_i)} f(v) \geq 2$ for $i = 1, 2, \dots, n$. Since $f(s_1) + f(s) + \sum_{j=1}^m f(c_j) \geq 2$, we obtain $\gamma_R(G) = w(f) \geq 4n + 2$. On the other hand,

$$(V(G) \setminus \{s, u_i, b_i : i = 1, 2, \dots, n\}, \emptyset, \{s, u_i, b_i : i = 1, 2, \dots, n\})$$

is an RDF for G of weight $4n + 2$. Hence $\gamma_R(G) = 4n + 2$.

We show that \mathcal{C} is satisfiable if and only if $r(G) = 1$. Assume that \mathcal{C} is satisfiable. Let $t: U \rightarrow \{T, F\}$ be a satisfying truth assignment for \mathcal{C} . We construct a subset D of vertices of G as follows. If $t(u_i) = T$, then we put the vertices u_i and b_i in D ; if $t(u_i) = F$, then put the vertices \bar{u}_i and d_i in D . Clearly, $|D| = 4n$. Now $(V(G) - (D \cup \{s, s_1\}), \{s, s_1\}, D)$ is a $\gamma_R(G)$ -function, while $(V(G) - (D \cup \{s\}), \{s\}, D)$ is an RDF for $G + xs_1$, where $x \in D$. Thus $r_R(G) = 1$.

Conversely, assume that $r_R(G) = 1$. Thus there is an edge $e \in E(\bar{G})$ such that $\gamma_R(G + e) < 4n + 2$. Let $g = (V_0^g, V_1^g, V_2^g)$ be a $\gamma_R(G + e)$ -function. Suppose that $\sum_{v \in V(H_i)} g(v) \leq 3$, for some $i \in \{1, 2, \dots, n\}$. Then there is a vertex $v \in V(H_i) \cap V_0^g$ such that v is dominated by some vertex of $V_2^g - V(H_i)$ and $\gamma_R(H_i - v) \leq 3$. This means that there is a vertex of degree at least 5 in $V(H_i)$, a contradiction. Thus $\sum_{v \in V(H_i)} g(v) = 4$, for some $i \in \{1, 2, \dots, n\}$. If $\{u_i, \bar{u}_i\} \subseteq V_2^g$ for some i , then a_i and f_i are not dominated by g , a contradiction. Thus $|\{u_i, \bar{u}_i\} \cap V_2^g| \leq 1$. Since $\sum_{v \in V(H_i)} g(v) = 4$ for some $i \in \{1, 2, \dots, n\}$ and $w(g) \leq 4n + 1$, we obtain $w(g) = 4n + 1$, $\sum_{j=1}^m g(c_j) = 0$ and $g(s) \neq 2$. Thus any vertex of $\{c_1, c_2, \dots, c_m\}$ is dominated by a vertex of $\{u_i, \bar{u}_i\}$, for some $i \in \{1, 2, \dots, n\}$.

Let $t: U \rightarrow \{T, F\}$ be a mapping defined by $t(u_i) = T$ if $u_i \in S$, and $t(u_i) = F$ if $\bar{u}_i \in S$. For each $j \in \{1, 2, \dots, m\}$, there is an integer $i \in \{1, 2, \dots, n\}$ such

that c_j is dominated by $V_2^g \cap \{u_i, \bar{u}_i\}$. Assume that $u_i \in V_2^g$ and c_j is dominated by u_i . By the construction of G , the literal u_i is in the clause C_j . Then $t(u_i) = T$, which implies that the clause C_j is satisfied by t . Next assume that $\bar{u}_i \in V_2^g$ and c_j is dominated by \bar{u}_i . By the construction of G , the literal \bar{u}_i is in the clause C_j . Then $t(u_i) = F$. Thus t assigns \bar{u}_i the truth value T , that is, t satisfies the clause C_j . Hence \mathcal{C} is satisfiable. \blacksquare

4 Graphs with large Roman reinforcement number

In this section we characterize graphs with Roman reinforcement number equal to maximum degree.

Observation 6 *If E' is a minimum set of edges such that $\gamma_R(G + E') < \gamma_R(G)$, then $\gamma_R(G + E') = \gamma_R(G) - 1$.*

Theorem 7 *Let G be a connected graph with $\Delta \geq 2$ and $\gamma_R(G) \geq 4$. We have $r_R(G) = \Delta$ if and only if there is a $\gamma_R(G)$ -function $f = (V_0, V_1, V_2)$ such that $V_1 = \emptyset$ and V_2 is an efficient dominating set of G every vertex of which has maximum degree.*

Proof. Assume that $r_R(G) = \Delta$. Suppose that there is a $\gamma_R(G)$ -function $h = (V_0^h, V_1^h, V_2^h)$ such that $V_1^h \neq \emptyset$. If $V_2^h = \emptyset$, then $\gamma_R(G) = n$. Using Proposition 1 we get $n = \gamma_R(G) \leq n - \Delta + 1$, implying that $\Delta \leq 1$, a contradiction. Thus $V_2^h \neq \emptyset$. Clearly, $r_R(G) = 1 < \Delta$, a contradiction. Thus for any $\gamma_R(G)$ -function $f = (V_0^f, V_1^f, V_2^f)$ we have $V_1^f = \emptyset$. We now get

$$n = |V(G)| = \left| \bigcup_{v \in V_2^f} N_G[v] \right| \leq \sum_{v \in V_2^f} (d_G(v) + 1) \leq |V_2^f| \cdot (\Delta + 1) = \frac{\gamma_R(G)}{2} \cdot (\Delta + 1).$$

Equivalently, $2n/\gamma_R(G) \leq \Delta + 1$. Since $r_R(G) = \Delta$, by Theorem 4 we get $\Delta + 1 \leq 2n/\gamma_R(G)$. Consequently, $\Delta + 1 = 2n/\gamma_R(G)$. This implies that G has a $\gamma_R(G)$ -function $f = (V_0, V_1, V_2)$ such that $V_1 = \emptyset$ and V_2 is an efficient dominating set of G . Moreover, equality holds throughout the above inequality chain. Thus $d_G(v) = \Delta$ for every vertex $v \in V_2$.

Now assume that G has a $\gamma_R(G)$ -function $f = (V_0, V_1, V_2)$ such that $V_1 = \emptyset$ and V_2 is an efficient dominating set of G every vertex of which has maximum degree. Then $V(G) = N_G[V_2]$ and $n = |V(G)| = |N_G[V_2]| = (\Delta + 1)|V_2|$. Let E' be a minimum set of edges that need to be added to G in order to decrease the Roman domination number. Then $r_R(G) = |E'|$. Let $H = G + E'$, and let $g = (V_0^g, V_1^g, V_2^g)$ be a $\gamma_R(H)$ -function. We have $|V_1^g| \geq 1$, $V(G) \setminus V_1^g = N_H[V_2^g]$

and $|N_H[V_2^g]| = |V| - |V_1^g| = n - |V_1^g| = (\Delta + 1)|V_2| - |V_1^g|$. Let $V' = V(G) \setminus N_G(V_2^g)$ denote the set of vertices not dominated by the set V_2^g in G . Then $|V'| \leq |E'|$. Since $2|V_2^g| + |V_1^g| \leq 2|V_2| - 1$, we obtain $|V_2^g| \leq |V_2| - (1 + |V_1^g|)/2$. We now get $|N_H[V_2^g]| = |N_G[V_2^g]| + |V'| \leq (\Delta + 1)|V_2^g| + |E'| \leq (\Delta + 1)(|V_2| - (|V_1^g| + 1)/2) + |E'|$. Hence, $(\Delta + 1)|V_2| - |V_1^g| = |N_H[V_2^g]| \leq (\Delta + 1)(|V_2| - (1 + |V_1^g|)/2) + |E'|$. Consequently,

$$r_R(G) = |E'| \geq \frac{\Delta + 1}{2}(1 + |V_1^g|) - |V_1^g| = \frac{\Delta + 1}{2} + \frac{\Delta - 1}{2}|V_1^g| \geq \Delta.$$

By Theorem 3 we have $r_R(G) = \Delta$. ■

Theorem 8 *Let G be a connected graph with $\Delta \geq k \geq 2$ and $\gamma_R(G) \geq 4$. If $r_R(G) = k$, then there is a $\gamma_R(G)$ -function $f = (V_0, V_1, V_2)$ such that $V_1 = \emptyset$, every vertex of V_2 has at least k private neighbors in V_0 , and some vertex of V_2 has precisely k private neighbors in V_0 .*

Proof. The result we prove by induction on $t = \Delta - k$. For $t = 0$, the result follows from Theorem 7. Assume that the result is correct for $0 < t' < t$. Let G be a connected graph such that $\gamma_R(G) \geq 4$ and $r_R(G) = k$. Let $h = (V_0^h, V_1^h, V_2^h)$ be a $\gamma_R(G)$ -function. Assume that $V_1^h \neq \emptyset$. If $V_2^h = \emptyset$, then $\gamma_R(G) = n$. By Proposition 1 we have $n = \gamma_R(G) \leq n - \Delta + 1$, implying that $\Delta \leq 1$, a contradiction. Thus $V_2^h \neq \emptyset$. Clearly, $r_R(G) = 1 < k$, a contradiction. Thus $V_1^h = \emptyset$. Assume that some vertex $x \in V_2^h$ has less than k private neighbors in V_0^h . Since $\gamma_R(G) \geq 4$, we choose a vertex $y \in V_2$, connect any private neighbor of x to y , and replace $h(x)$ by 1, to deduce that $r_R(G) \leq k - 1$, a contradiction. Thus for any $\gamma_R(G)$ -function $f = (V_0, V_1, V_2)$ such that $V_1 = \emptyset$, any vertex of V_2^f has at least k private neighbors in V_0^f . If every vertex of V_2 has at least $k + 1$ private neighbors in V_0 , then by the inductive hypothesis we have $r_R(G) > k$, a contradiction. Thus there is a vertex of V_2 with precisely k private neighbors in V_0 . ■

Proposition 9 *Let G be a connected graph with $\Delta \geq k \geq 2$ and $\gamma_R(G) \geq 4$. Assume that there is a $\gamma_R(G)$ -function $f = (V_0, V_1, V_2)$ such that $V_1 = \emptyset$, every vertex of V_2 has at least k private neighbors in V_0 , and there is a vertex of V_2 with precisely k private neighbors in V_0 . If $E' \subseteq E(\overline{G})$ and g is a $\gamma_R(G + E')$ -function such that $|V_1^g| > 1$, then $|E'| \geq k$.*

Proof. Assume that G has a $\gamma_R(G)$ -function $f = (V_0, V_1, V_2)$ such that $V_1 = \emptyset$, every vertex of V_2 has at least k private neighbors in V_0 , and some vertex of V_2 has precisely k private neighbors in V_0 . Consider a vertex $v_1 \in V_2$ such that $d_G(v_1)$ is maximum, and let $G_1 = N_G[v_1] - V_0$. Next consider a vertex $v_2 \in V_2$

such that $d_{G-G_1}(v_2)$ is maximum, and let $G_2 = N_{G-G_1}[v_2] - V_0$. Continuing this process we obtain graphs $G_1, G_2, \dots, G_{|V_2|}$ such that $V(G)$ is partitioned into $V(G_1), V(G_2), \dots, V(G_{|V_2|})$. Since any vertex of V_2 has at least k private neighbors in V_0 , we have $k+1 \leq |V(G_i)| \leq \Delta+1$, for $i = 1, 2, \dots, r$. Furthermore, by assumption there is an integer j such that $|V(G_j)| = k+1$. For $i = k+1, k+2, \dots, \Delta+1$, let A_i be the set of all vertices v_i such that $|V(G_i)| = i$. Then

$$n = |V(G)| = |A_{k+1}|(k+1) + |A_{k+2}|(k+2) + \dots + |A_{\Delta+1}|(\Delta+1).$$

Let $E' \subseteq E(\overline{G})$, and let g be a $\gamma_R(G + E')$ -function such that $|V_1^g| > 1$. Suppose to the contrary that $|E'| \leq k-1$. Then $V(G) - V_1^g = N_H[V_2^g]$ and $|N_H[V_2^g]| = |V| - |V_1^g| = n - |V_1^g|$. Assume that $|V_1^g| = 2$. Then $|V_2^g| \leq |V_2^f| - 2$ and we get

$$\begin{aligned} n &= 2 + |N_H[V_2^g]| \\ &\leq 2 + \left((|A_{k+1}| - 2)(k+1) + |A_{k+2}|(k+2) + \dots + |A_{k+t+1}|(k+t+1) + |E'| \right) \\ &\leq 2 + n - 2(k+1) + |E'| \\ &\leq 2 + n - 2(k+1) + k - 1 \\ &= n - k - 1. \end{aligned}$$

This implies that $k \leq 1$, a contradiction. If $|V_1^g| \geq 3$, then similarly we obtain a contradiction. \blacksquare

It is an open problem whether the converse of Theorem 8 is true.

5 Trees

A graph G is called a Roman graph if $\gamma_R(G) = 2\gamma(G)$. The authors of [4] observed that a graph G is Roman if and only if there is a $\gamma_R(G)$ -function $f = (V_0, V_1, V_2)$ such that $V_1 = \emptyset$. Henning [10] characterized all Roman trees.

A graph G is called a strong Roman graph if $V_1 = \emptyset$ for every $\gamma_R(G)$ -function $f = (V_0, V_1, V_2)$. We characterize all strong Roman trees.

Let us observe that for any graph G , the inequality $r_R(G) > 1$ holds if and only if G is a strong Roman graph. The following result is a direct consequence of Theorem 2.

Theorem 10 *Let G be a connected graph of order $n \geq 3$. Then $r_R(G) > 1$ if and only if for every $\gamma_R(G)$ -function $f = (V_0, V_1, V_2)$ we have $V_1 = \emptyset$.*

Let \mathcal{F} denote the family of rooted trees on at least four vertices such that every leaf other than the root is at distance two from the root, and every child of the root is a strong support vertex. For the purpose of characterizing all strong Roman trees, that is, trees T such that $r_R(T) > 1$, we introduce a family \mathcal{T}

of trees $T = T_k$ that can be obtained as follows. Let T_1 be any star. If k is a positive integer, then T_{k+1} can be obtained recursively from T_k by one of the following operations.

- Operation \mathcal{O}_1 : Attach a star on at least four vertices by joining the central vertex to a support vertex of T_k , say x , such that $\gamma_R(T_k - x) > \gamma_R(T_k)$.
- Operation \mathcal{O}_2 : Attach a tree of the family \mathcal{F} by joining the root to any vertex of T_k .
- Operation \mathcal{O}_3 : Attach a path P_3 by joining one of its leaves to a vertex of T_k to which every $\gamma_R(T_k)$ -function assigns 0.

We now prove that if a tree belongs to the family \mathcal{T} , then it is strong Roman.

Lemma 11 *If $T \in \mathcal{T}$, then T is a strong Roman tree.*

Proof. We use the induction on the number k of operations performed to construct the tree T . If $T = T_1$ is a star, then it is easy to observe that there is only one $\gamma_R(T)$ -function (the weight of the central vertex is 2 and the weight of each leaf is 0). Thus all stars are strong Roman. Let k be a positive integer. Assume that the result is true for every tree $T' = T_k$ of the family \mathcal{T} constructed by $k - 1$ operations. Let $T = T_{k+1}$ be a tree of the family \mathcal{T} constructed by k operations.

First assume that T is obtained from T' by operation \mathcal{O}_1 . The central vertex of the attached star we denote by v . Let f' be any $\gamma_R(T')$ -function. It is easy to observe that we can extend it to an RDF for the tree T by assigning 2 to v and 0 to the leaves adjacent to v . Thus $\gamma_R(T) \leq \gamma_R(T') + 2$. Now let f be any $\gamma_R(T)$ -function. Clearly, $f(v) = 2$ and the weight of each leaf adjacent to v is 0. Suppose that $V_1^f \neq \emptyset$. If $f(x) \neq 0$, then $f|_{V(T')}$ is a $\gamma_R(T')$ -function with $V_1 \neq \emptyset$, a contradiction. Now assume that $f(x) = 0$. Let us observe that $f|_{V(T'-x)}$ is an RDF for the graph $T' - x$. Therefore $\gamma_R(T' - x) \leq \gamma_R(T) - 2$. We now get $\gamma_R(T' - x) \leq \gamma_R(T) - 2 \leq \gamma_R(T')$, a contradiction. This implies that for every $\gamma_R(T)$ -function we have $V_1 = \emptyset$, that is, the tree T is strong Roman.

Now assume that T is obtained from T' by operation \mathcal{O}_2 . The root of the attached tree we denote by x . It is easy to observe that $\gamma_R(T) \leq \gamma_R(T') + 2d_T(x) - 2$. Let f be any $\gamma_R(T)$ -function. Let us observe that the weight of every neighbor of x in the tree of the family \mathcal{F} is 2, while the weight of each other vertex of that tree is 0. Suppose that $V_1^f \neq \emptyset$. Observe that $f|_{V(T')}$ is a $\gamma_R(T')$ -function with $V_1 \neq \emptyset$, a contradiction. Thus T is a strong Roman tree.

Now assume that T is obtained from T' by operation \mathcal{O}_3 . The vertex to which is attached P_3 we denote by x . Let $v_1v_2v_3$ be the attached path. Let v_1 be joined to x . It is easy to see that $\gamma_R(T) \leq \gamma_R(T') + 2$. Let f be any $\gamma_R(T)$ -function. Let us observe that $f(v_1) \neq 1$. Suppose that $V_1^f \neq \emptyset$. If $f(v_1) = 2$, then notice that $f(v_2) = 0$ and $f(v_3) = 1$. Now modifying $f|_{V(T')}$ by letting $f(x) = 1$, we

get a $\gamma_R(T')$ -function with $V_1 \neq \emptyset$, a contradiction. Now assume that $f(v_1) = 0$. If $f(v_2) \neq 2$, then $f(x) = 2$ as the vertex v_1 has to be dominated. Observe that $f|_{V(T')}$ is a $\gamma_R(T')$ -function with $f(x) = 2$, a contradiction. Thus $f(v_2) = 2$. Now $f|_{V(T')}$ is a $\gamma_R(T')$ -function with $V_1 \neq \emptyset$, a contradiction. Thus the tree T is strong Roman. ■

We now prove that if a tree T is strong Roman, then it belongs to the family \mathcal{T} .

Lemma 12 *If T is a strong Roman tree, then $T \in \mathcal{T}$.*

Proof. If $\text{diam}(T) \leq 1$, then $T \in \{P_1, P_2\}$. By assigning 1 to each vertex, we get $\gamma_R(T)$ -functions with $V_1 \neq \emptyset$. Thus the paths P_1 and P_2 are not strong Roman. If $\text{diam}(T) = 2$, then T is a star. We have $T \in \mathcal{T}$. Now assume that $\text{diam}(T) = 3$. Thus T is a double star with central vertices x and y . Since T is a strong Roman tree, we have $d_T(x) \geq 4$ and $d_T(y) \geq 4$. Let T_x be the component of $T - xy$ which contains the vertex x . The tree T_x is a star, thus $T_x \in \mathcal{T}$. Clearly, $\gamma_R(T_x - x) > \gamma_R(T_x)$. The tree T can be obtained from T_x by operation \mathcal{O}_1 . Thus $T \in \mathcal{T}$.

The result we obtain by the induction on the number n . Assume that the lemma is true for every tree T' of order $n' < n$. We root T at a vertex r of maximum eccentricity $\text{diam}(T)$. Let t be a leaf at maximum distance from r , v be the parent of t , u be the parent of v , and w be the parent of u in the rooted tree. By T_x we denote the subtree induced by a vertex x and its descendants in the rooted tree T .

First assume that some child of u , say x , is a leaf. If $d_T(v) \leq 3$, then let f be any $\gamma_R(T)$ -function. Clearly, $f(v) = f(u) = 2$. Let us observe that by letting v have the weight 0 and each child of v the weight 1, we get a $\gamma_R(T)$ -function with $V_1 \neq \emptyset$, a contradiction. Thus $d_T(v) \geq 4$. Let $T' = T - T_v$. Let f be any $\gamma_R(T)$ -function. Notice that $f(u) = f(v) = 2$ and the weight of each child of v is 0. It is easy to observe that $f|_{V(T')}$ is an RDF for the tree T' . Therefore $\gamma_R(T') \leq \gamma_R(T) - 2$. Now let f' be any $\gamma_R(T')$ -function. Suppose that $V_1^{f'} \neq \emptyset$. It is easy to see that extending it to a $\gamma_R(T)$ -function by assigning 2 to v and 0 to each child of v gives a $\gamma_R(T)$ -function with $V_1 \neq \emptyset$, a contradiction. Therefore T' is a strong Roman tree. By the inductive hypothesis we have $T' \in \mathcal{T}$. Suppose that $\gamma_R(T' - u) \leq \gamma_R(T')$. We get $\gamma_R(T - u) = \gamma_R(T_v \cup T' - u) = \gamma_R(T' - u) + \gamma_R(T_v) \leq \gamma_R(T') + 2 \leq \gamma_R(T)$. Let us observe that every $\gamma_R(T - u)$ -function is a $\gamma_R(T)$ -function as the weight of v is 2. Obviously, every RDF for $T - u$ assigns 1 to the vertex x . Thus there is a $\gamma_R(T)$ -function with $V_1 \neq \emptyset$, a contradiction. Therefore $\gamma_R(T' - u) > \gamma_R(T')$. The tree T can be obtained from T' by operation \mathcal{O}_1 . Thus $T \in \mathcal{T}$.

Now assume that all children of u are support vertices. First assume that $d_T(u) \geq 3$. Assume that some child of u , say x , is a weak support vertex. The leaf adjacent to x we denote by y . Let f be any $\gamma_R(T)$ -function. Clearly,

$f(v) = f(x) = 2$ and $f(y) = 0$. Let us observe that by letting $f(x) = f(y) = 1$, we get a $\gamma_R(T)$ -function with $V_1 \neq \emptyset$, a contradiction. Thus all children of u are strong support vertices. Let $T' = T - T_u$. It is easy to observe that $\gamma_R(T') \leq \gamma_R(T) - 2d_T(u) + 2$. Let f' be any $\gamma_R(T')$ -function. Suppose that $V_1^{f'} \neq \emptyset$. It is easy to see that extending it to a $\gamma_R(T)$ -function by assigning 2 to each child of u and 0 to the remaining vertices of T_u gives a $\gamma_R(T)$ -function with $V_1 \neq \emptyset$, a contradiction. Therefore T' is a strong Roman tree. By the inductive hypothesis we have $T' \in \mathcal{T}$. The tree T can be obtained from T' by operation \mathcal{O}_2 . Thus $T \in \mathcal{T}$.

Now assume that $d_T(u) = 2$. Let $T' = T - T_u$. It is easy to observe that $\gamma_R(T') \leq \gamma_R(T) - 2$. We prove that for every $\gamma_R(T')$ -function f' we have $V_1^{f'} = \emptyset$ and $f'(w) = 0$. Let f' be any $\gamma_R(T')$ -function. Suppose that $V_1^{f'} \neq \emptyset$. Extending f' by assigning 2 to v and 0 to u and t , we get a $\gamma_R(T)$ -function with $V_1 \neq \emptyset$, a contradiction. Now suppose that $f'(w) \neq 0$, that is, $f'(w) = 2$. Extending f' by letting $f'(u) = 0$ and $f'(v) = f'(t) = 1$, we get a $\gamma_R(T)$ -function with $V_1 \neq \emptyset$, a contradiction. Thus every $\gamma_R(T')$ -function assigns 0 to the vertex w . The tree T' is strong Roman. By the inductive hypothesis we have $T' \in \mathcal{T}$, and T can be obtained from T' by operation \mathcal{O}_3 . Thus $T \in \mathcal{T}$. ■

As an immediate consequence of Lemmas 11 and 12, we get the following characterization of strong Roman trees.

Theorem 13 *A tree T is strong Roman if and only if $T \in \mathcal{T}$.*

Corollary 14 *Let T be a tree. Then $r_R(T) > 1$ if and only if $T \in \mathcal{T}$.*

We next characterize all trees T with $r_R(T) = \Delta$. For this purpose we introduce families \mathcal{T}_r ($r \geq 2$) of trees $T = T_k$ that can be obtained as follows. In \mathcal{T}_r , let T_1 be the tree obtained from two stars $K_{1,r}$ by joining them through leaves. If k is a positive integer, then T_{k+1} can be obtained recursively from T_k by attaching a star $K_{1,r}$ by joining one of its leaves to a vertex of T_k of degree less than $\Delta(T_k)$.

Notice that for $r \geq 3$, the family \mathcal{T}_r is a subclass of the family \mathcal{T} , and thus every tree of \mathcal{T}_r is strong Roman.

Theorem 15 *Let T be a tree. We have $r_R(T) = \Delta$ if and only if $T \in \mathcal{T}_\Delta$.*

Proof. First assume that T is a tree of the family \mathcal{T}_Δ . If $T = T_1$, then it is not difficult to observe that $r_R(T) = \Delta$. Let k be a positive integer. Assume that the result is true for every tree $T' = T_k$ of the family \mathcal{T}_Δ constructed by $k - 1$ operations. Let $T = T_{k+1}$ be a tree of the family \mathcal{T}_Δ constructed by k operations. As noted, T is a strong Roman tree, and it can be seen that $\gamma_R(T) = \gamma_R(T') + 2$ and $\gamma(T) = \gamma(T') + 1$. By Theorem 7, there is a $\gamma_R(T)$ -function $f = (V_0, V_1, V_2)$

such that $V_1 = \emptyset$, every vertex of V_2 has maximum degree, and V_2 is an efficient dominating set for T . Then we extend f by assigning 2 to the central vertex of the star and 0 to every leaf of the star to obtain a $\gamma_R(T)$ -function $g = (V_0^g, V_1^g, V_2^g)$ such that $V_1^g = \emptyset$, every vertex of V_2^g has maximum degree, and V_2^g is an efficient dominating set for T . By Theorem 7 we have $r_R(T) = \Delta$.

Now assume that T is a tree with $r_R(T) = \Delta$. Clearly, $\Delta \geq 2$. We proceed by induction on the domination number $\gamma(T)$. Clearly, $\gamma(T) \geq 2$ as $\Delta \geq 2$. If $\text{diam}(T) = 3$, then T is a double star and it can be seen that $r_R(T) < \Delta$, a contradiction. Assume that $\text{diam}(T) = 4$. Let a and b be two support vertices at distance two, and let f be a $\gamma_R(T)$ -function. Clearly, $f(a) = f(b) = 2$, as otherwise $V_1^f \neq \emptyset$ and by Theorem 10 we have $r_R(T) = 1$, a contradiction. Now $(V_0^f \cup N_T(b), \{b\}, V_2^f)$ is an RDF for $T + ab_1 + ab_2 + \dots + ab_l$, where $N_T(b) \setminus N_T(a) = \{b_1, b_2, \dots, b_l\}$, a contradiction to $r_R(T) = \Delta$.

Now assume that $\text{diam}(T) = 5$. Let $a_1a_2a_3a_4a_5$ be a diametrical path, and let f_1 be a $\gamma_R(T)$ -function. Clearly, $f_1(a_2) = f_1(a_5) = 2$, as otherwise $V_1^{f_1} \neq \emptyset$ and by Theorem 10 we have $r_R(T) = 1$, a contradiction. Furthermore, $f_1(a_4) = 0$. Clearly, a_4 is not a support vertex, as otherwise $V_1^{f_1} \neq \emptyset$, a contradiction. If $d_T(a_4) > 2$, then a_4 has a neighbor $u \neq a_5$ which is a support vertex, and $d(u, a_1) = 4$. Similarly as earlier we observe that $f_1(u) = 2$. Now $(V_0^{f_1} \cup N_T(u), \{u\}, V_2^{f_1})$ is an RDF for $T + a_5u_1 + a_5u_2 + \dots + a_5u_l$, where $N_T(u) \setminus N_T(a_5) = \{u_1, u_2, \dots, u_l\}$, a contradiction to $r_R(T) = \Delta$. We deduce that $d_T(a_4) = 2$. Similarly, $d_T(a_3) = 2$. Let T' be the component of $T - a_3a_4$, which contains a_3 . Then $T' = K_{1,\Delta}$, and thus $T' \in \mathcal{T}_\Delta$. The tree T can be obtained from T' by attaching a star $K_{1,r}$ by joining one of its leaves to a vertex of T' of degree less than $\Delta(T')$. Thus $T \in \mathcal{T}_\Delta$.

Suppose that the result is true for all strong Roman trees with domination number less than m , where $m \geq 2$, and let T be a strong Roman tree with $\gamma(T) = m$. Then $\text{diam}(T) \geq 6$. Let T be rooted at the end-vertex x of a longest path P . Let x_{d-2} be the vertex at distance $\text{diam}(T) - 2$ from x on P , and let x_{d-1} be the child of x_{d-2} on P . Let x_{d-3} denote the parent of x_{d-2} . Clearly, $d_T(x_{d-1}) = \Delta$. If $d_T(x_{d-2}) \geq 3$, then similarly as earlier we get a contradiction. Thus $d_T(x_{d-2}) = 2$. Let $T' = T - T_{x_{d-2}}$. It can be seen that $\gamma(T) = \gamma(T') + 1$ and $\gamma_R(T) = \gamma_R(T') + 2$. If $r_R(T') < \Delta$, then we can easily obtain $r_R(T) < \Delta$, a contradiction. Thus $r_R(T') = \Delta$. By the inductive hypothesis we have $T' \in \mathcal{T}_\Delta$. Now T can be obtained from T' by attaching a star $K_{1,r}$ by joining one of its leaves to a vertex of T' of degree less than $\Delta(T')$. Thus $T \in \mathcal{T}_\Delta$. ■

We end with a realization result. For any two integers k and m with $m > k > 2$, we join a leaf of a star $K_{1,m}$ to a leaf of a tree $T' \in \mathcal{T}_k$ to obtain a tree T . Then it is straightforward to see that $\Delta(T) = m$ and $r_R(T) = k$. Thus we obtain the following result.

Theorem 16 *For any $2 < k < m$, there is a graph G with $\Delta(G) = m$ and*

$$r_R(G) = k.$$

Open problem

Problem 17 *Is the converse of Theorem 8 true?*

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