# Double bondage in graphs

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**Abstract**

Let $G=(V,E)$ be a graph with *vertex set* $V$ and *edge set* $E$. A vertex $v \in V(G)$ is said to *dominate* itself and all vertices in its *neighborhood* $N_G(v)=\{u\colon uv \in E(G)\}$. A set $D \subseteq V(G)$ is a *double dominating set* of $G$ if every vertex $v \in V(G)$ is dominated by at least two vertices of $D$. The *double domination number* $\gamma_d(G)$ equals the minimum cardinality of a double dominating set of $G$. Let $E' \subseteq E$ be a subset of the set of edges, and let $G-E'=(V,E-E')$ be the graph obtained from $G$ by removing the edges of $E'$. The *double bondage number* $b_d(G)$ equals the minimum cardinality of a set $E' \subseteq E$ such that the graph $G-E'$ does not contain an *isolated vertex* (that is, a vertex $u$ with $N_G(u)=\emptyset$) and $\gamma_d(G-E')>\gamma_d(G)$. If for every subset $E' \subseteq E$, either $\gamma_d(G-E')=\gamma_d(G)$ or $G-E'$ contains an isolated vertex, then we define $b_d(G)=0$, and we say that $G$ is a $\gamma_d$-*strongly stable graph*. We present several basic properties of double bondage in graphs and we determine the double bondage numbers of several classes of graphs. We also characterize the class of trees $T$ for which $b_d(T)=1$.
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## 1 Introduction

Let $G=(V,E)$ be a graph of *order* $n=|V|$. The *(open) neighborhood* of a vertex $v \in V(G)$ is the set $N_G(v)=\{u \in V(G)\colon uv \in E(G)\}$; vertices in $N_G(v)$ are called the *neighbors* of $v$. The *degree* of a vertex $v$ is $d_G(v)$

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$= |N_G(v)|$. An *isolated vertex* is a vertex $v$ for which $d_G(v) = 0$. Let $\delta(G)$ be equal to the minimum degree of a vertex $v \in V(G)$. A set $S \subseteq V(G)$ is *independent* if no two vertices in $S$ are adjacent.

Let $E' \subseteq E(G)$ be a subset of the set edges of $G$. Let $G' = (V, E - E')$ be the graph obtained from $G$ by removing the edges of $E'$. Since $V(G') = V(G)$, we say that $G'$ is a *spanning subgraph* of $G$.

A *leaf* is a vertex $v$ with $d_G(v) = 1$, and a *support vertex* is a vertex having a neighbor that is a leaf. A support vertex is called *strong* if it has two or more leaf neighbors, otherwise it is called a *weak* support vertex.

Let $T$ be a tree and let $uv$ be an edge of $T$. Removing the edge $uv$ from $T$ produces a graph consisting of two subtrees, $T_u$ and $T_v$. In this case, we say that the vertex $u$ is *adjacent to* the subtree $T_v$, and conversely, the vertex $v$ is adjacent to the subtree $T_u$. Let $P_n$ denote the *path* of order $n$ (and length $n-1$). A vertex $u$ in a tree $T$ is adjacent to a path $P_n$ if there is an edge $uv$ such that the subtree $T_v$ is isomorphic to $P_n$, in which $v$ is a leaf.

A *star*, denoted by $K_{1,m}$, is a tree of order $m+1$ having exactly one vertex of degree greater than one. Note that by the definition, a star has at least three vertices. The process of *subdividing an edge*, say $uv$, in a graph $G$ consists of removing the edge $uv$, adding a new vertex, say $x$, and adding two new edges $ux$ and $xv$. A *subdivided star*, denoted by $S(K_{1,m})$, is a graph obtained from a star $K_{1,m}$ by subdividing each one of its $m$ edges.

Let $C_n$ denote a *cycle* of order $n$ and let $K_n$ denote a *complete graph* of order $n$. The *join* of two graphs $G$ and $H$ is the graph $G+H$ obtained from the disjoint union of $G$ and $H$ by adding an edge between each vertex of $G$ and each vertex of $H$. A *wheel* is a graph of the form $W_n = K_1 + C_{n-1}$. A *complete bipartite graph* $K_{p,q} = \overline{K_p} + \overline{K_q}$ has two *independent sets* of orders $p$ and $q$, called its *partite sets*.

A vertex $v \in V(G)$ is said to *dominate* itself and all of its neighbors. A set $D \subseteq V(G)$ is called a *dominating set* of $G$ if every vertex $v \in V(G)$ is dominated by at least one vertex of $D$, and it is a *double dominating set*, abbreviated DDS, of $G$ if every vertex $v \in V(G)$ is dominated by at least two vertices of $D$. The *domination number* $\gamma(G)$ and *double domination number* $\gamma_d(G)$ are equal to the minimum cardinalities of a dominating set and double dominating set of $G$, respectively. A dominating (double dominating) set of $G$ of minimum cardinality is called a $\gamma(G)$-set ($\gamma_d(G)$-set). Note that an isolated vertex cannot be dominated by two vertices. Therefore, while considering double domination, we always assume that a graph has no isolated vertices.

Double domination in graphs was introduced by Harary and Haynes in [11] and is further studied for example in [1–5, 8–10, 17–21]. For a comprehensive survey of domination in graphs, see Haynes, Hedetniemi, and Slater [14, 15].

The *bondage number* $b(G)$ of a graph $G$ equals the minimum cardinality of a set of edges $E' \subseteq E$ such that $\gamma(G - E') > \gamma(G)$. If for every $E' \subseteq E$, $\gamma(G - E') = \gamma(G)$, then we define $b(G) = 0$ and we say that $G$ is $\gamma$-*strongly stable*. Bondage in graphs was introduced by Fink, Jacobson, Kinch, and Roberts in [7] and is further studied for example in [6, 12, 13, 16, 22, 24, 25].

Yogeesha and Soner introduced the *double bondage number* $b_d(G)$ in [26], as the minimum cardinality of a set of edges $E' \subseteq E$ such that $\delta(G - E') \geq 1$ and $\gamma_d(G - E') > \gamma_d(G)$. If for every $E' \subseteq E$, either $\gamma_d(G - E') = \gamma_d(G)$ or $\delta(G - E') = 0$, then we define $b_d(G) = 0$ and we say that $G$ is $\gamma_d$-*strongly stable*.

In a distributed network, some vertices act as resource centers, or servers, while other vertices are clients. If a set $D$ of servers is a dominating set, then every client in $V(G) \setminus D$ has direct (one hop) access to at least one server. Double dominating sets represent a higher level of service, since every client has guaranteed access to at least two servers. In addition, every server in $D$ has access to at least one other server in $D$, as a backup. The double bondage number of a graph, therefore, provides an estimate of the cost of continuing to provide double server access in the event that certain communication links fail.

In Section 2 we discuss some basic properties of double domination and double bondage in graphs, we determine double domination numbers and double bondage numbers for several classes of graphs, and we characterize the class of $\gamma_d$-strongly stable graphs. In Section 3 we characterize the class of trees T for which $b_d(T) = 1$.

## 2 Basic properties of double domination and double bondage

In this and the following sections we assume that all graphs $G$ are connected and have order $n \geq 2$. We begin with the following observations, the majority of which were given in [10].

**Observation 1** *If $v$ is a leaf of a graph $G$, then $v$ is an element of every $\gamma_d(G)$-set.*

**Observation 2** *If $v$ is a support vertex of a graph $G$, then $v$ is an element of every $\gamma_d(G)$-set.*

**Observation 3** *If $G' = (V, E - E')$ is a spanning subgraph of a graph $G = (V, E)$, then $\gamma_d(G') \geq \gamma_d(G)$.*

**Observation 4** *For $n \geq 2$ we have $\gamma_d(K_n) = 2$.*

**Observation 5** *For $n \geq 2$ we have $\gamma_d(P_n) = \lfloor (2n+4)/3 \rfloor$.*

**Observation 6** *For $n \geq 3$ we have $\gamma_d(C_n) = \lfloor (2n+2)/3 \rfloor$.*

**Observation 7** *For $n \geq 4$ we have $\gamma_d(W_n) = \lfloor (n+4)/3 \rfloor$.*

**Observation 8** *Let $p$ and $q$ be positive integers such that $p \leq q$. Then*

$$\gamma_d(K_{p,q}) = \begin{cases} q+1 & \text{if } p = 1; \\ 3 & \text{if } p = 2; \\ 4 & \text{if } p \geq 3. \end{cases}$$

Since graphs with an isolated vertex do not have a double dominating set, we do not consider removing edges that produce an isolated vertex.

The following three results are from [26].

**Proposition 9** *For $n \geq 2$ we have*

$$b_d(K_n) = \begin{cases} 0 & \text{if } n = 2; \\ \lfloor n/2 \rfloor & \text{if } n \geq 3. \end{cases}$$

**Proposition 10** *For $n \geq 2$ we have*

$$b_d(P_n) = \begin{cases} 0 & \text{if } n \leq 4; \\ 1 & \text{if } n \geq 5. \end{cases}$$

**Proposition 11** *For $n \geq 3$ we have*

$$b_d(C_n) = \begin{cases} 1 & \text{if } n \neq 3k+2; \\ 2 & \text{if } n = 3k+2. \end{cases}$$

To the previous three results we can add the following one.

**Proposition 12** *For $n \geq 4$ we have*

$$b_d(W_n) = \begin{cases} 1 & \text{if } n = 3k+1 \geq 7; \\ 2 & \text{if } n = 3k \text{ or } n \leq 5; \\ 3 & \text{if } n = 3k+2 \geq 8. \end{cases}$$

**Proof.** Let $E(W_n) = \{v_1v_2, v_1v_3, \ldots, v_1v_n, v_2v_3, v_3v_4, \ldots, v_{n-1}v_n, v_nv_2\}$. Using Proposition 9 we get $b_d(W_4) = b_d(K_4) = 2$. By Observation 7 we have $\gamma_d(W_5) = 3$. Let us observe that removing any one of the edges does not increase the double domination number as the vertices $v_1$, $v_2$, and $v_4$ still form a double dominating set. We have $\gamma_d(W_5 - v_1v_2 - v_2v_3) = 4 > 3 = \gamma_d(W_5)$. Therefore $b_d(W_5) = 2$.

Now let us assume that $n \geq 6$. First assume that $n = 3k+1$. Let us remove the edge $v_1v_n$. We find a relation between the numbers $\gamma_d(W_n$

$-v_1v_n$) and $\gamma_d(W_{n-2})$. Let us observe that there exists a $\gamma_d(W_n-v_1v_n)$-set that does not contain the vertex $v_n$. Let $D$ be such a set. The vertex $v_n$ has to be dominated twice, thus $v_2, v_{n-1} \in D$. We have $v_1 \in D$ or $v_{n-2} \in D$ as the vertex $v_{n-1}$ has to be dominated twice. Let us observe that $D\setminus\{v_{n-1}\}$ is a DDS of the graph $W_{n-2}$. Therefore $\gamma_d(W_{n-2}) \leq \gamma_d(W_n - v_1v_n) - 1$. Now we get $\gamma_d(W_n - v_1v_n) \geq \gamma_d(W_{n-2}) + 1 = \lfloor (n+2)/3 \rfloor + 1 = \lfloor (n+4)/3 \rfloor + 1 > \lfloor (n+4)/3 \rfloor = \gamma_d(W_n)$. Therefore $b_d(W_n) = 1$ if $n = 3k+1$.

Now assume that $n \neq 3k+1$. If we remove an edge incident with $v_1$, say $v_1v_2$, then we get $\gamma_d(W_n - v_1v_2) = \gamma_d(W_n)$ as we can construct a $\gamma_d(W_n)$-set that contains the vertices $v_3$ and $v_n$; such set is also a DDS of the graph $W_n - v_1v_2$. If we remove an edge non-incident with $v_1$, say $v_2v_3$, then we get $\gamma_d(W_n - v_2v_3) = \gamma_d(W_n)$ as we can construct a $\gamma_d(W_n)$-set that does not contain the vertices $v_2$ and $v_3$; such set is also a DDS of the graph $W_n - v_2v_3$. This implies that $b_d(W_n) \neq 1$.

First, assume that $n = 3k$. Let us remove the edges $v_{n-1}v_n$ and $v_nv_2$. We find a relation between the numbers $\gamma_d(W_n - v_{n-1}v_n - v_nv_2)$ and $\gamma_d(W_{n-1})$. Let $D$ be any $\gamma_d(W_n - v_{n-1}v_n - v_nv_2)$-set. By Observations 1 and 2 we have $v_1, v_n \in D$. The vertex $v_3$ has to be dominated twice, thus $v_3 \in D$ or $v_4 \in D$. Let us observe that $D \setminus \{v_n\}$ is a DDS of the graph $W_{n-1}$. Therefore $\gamma_d(W_{n-1}) \leq \gamma_d(W_n - v_{n-1}v_n - v_nv_2) - 1$. Using Observation 7 we get $\gamma_d(W_n - v_{n-1}v_n - v_nv_2) \geq \gamma_d(W_{n-1}) + 1 = \lfloor (n+3)/3 \rfloor + 1 = \lfloor (n+4)/3 \rfloor + 1 > \lfloor (n+4)/3 \rfloor = \gamma_d(W_n)$. Therefore $b_d(W_n) = 2$ if $n = 3k$.

Now let us assume that $n = 3k+2$. It is not very difficult to verify that removing any two edges does not increase the double domination number. This implies that $b_d(W_n) = 0$ or $b_d(W_n) \geq 3$. Let us remove the edges $v_{n-2}v_{n-1}$, $v_{n-1}v_n$, and $v_1v_n$. We find a relation between the numbers $\gamma_d(W_n - v_{n-2}v_{n-1} - v_{n-1}v_n - v_1v_n)$ and $\gamma_d(W_n)$. Let $D$ be any $\gamma_d(W_n - v_{n-2}v_{n-1} - v_{n-1}v_n - v_1v_n)$-set. By Observations 1 and 2 we have $v_1, v_2, v_{n-1}, v_n \in D$. Let us observe that $D \setminus \{v_{n-1}, v_n\}$ is a DDS of the graph $W_{n-2}$. Therefore $\gamma_d(W_{n-2}) \leq \gamma_d(W_n - v_{n-2}v_{n-1} - v_{n-1}v_n - v_1v_n) - 2$. Now we get $\gamma_d(W_n - v_{n-2}v_{n-1} - v_{n-1}v_n - v_1v_n) \geq \gamma_d(W_{n-2}) + 2 = \lfloor (n+2)/3 \rfloor + 2 = \lfloor (n+5)/3 \rfloor + 1 = \lfloor (n+4)/3 \rfloor + 1 > \lfloor (n+4)/3 \rfloor = \gamma_d(W_n)$. Therefore $b_d(W_n) = 3$ if $n = 3k+2$. ■

We now determine the double bondage numbers of complete bipartite graphs.

**Proposition 13** *Let $p$ and $q$ be positive integers such that $p \leq q$. Then*

$$b_d(K_{p,q}) = \begin{cases} 3 & \text{if } p = q = 3; \\ p-1 & \text{otherwise.} \end{cases}$$

**Proof.** Let $E(K_{p,q}) = \{a_ib_j \colon 1 \leq i \leq p \text{ and } 1 \leq j \leq q\}$. If $p = 1$, then $b_d(K_{p,q}) = 0 = p - 1$ as removing any edge produces an isolated vertex.

Now assume that $p = 2$. By Observation 8 we have $\gamma_d(K_{2,q}) = 3$. Let us observe that $\gamma_d(K_{2,q} - a_1b_1) = 4$ as the vertex $b_1$ has to belong to every DDS of the graph $K_{2,q} - a_1b_1$. Thus $b_d(K_{2,q}) = 1 = p - 1$.

Now let us assume that $p \geq 3$. By Observation 8 we have $\gamma_d(K_{3,q}) = 4$. If $q = 3$, then it is easy to verify that removing any two edges does not increase the double domination number. This implies that $b_d(K_{3,3}) = 0$ or $b_d(K_{3,3}) \geq 3$. Let us observe that $\gamma_d(K_{3,3} - a_1b_1 - a_1b_2 - a_2b_1) = 5 > 4 = \gamma_d(K_{3,3})$. Therefore $b_d(K_{3,3}) = 3$. Now assume that $q \geq 4$. Let us observe that removing any $p - 2$ edges does not increase the double domination number. We have $\gamma_d(K_{p,q} - a_1b_1 - a_2b_1 - \ldots - a_{p-1}b_1) = 5$ as the vertex $b_1$ has to belong to every DDS of the graph $K_{p,q} - a_1b_1 - a_2b_1 - \ldots - a_{p-1}b_1$. Therefore $b_d(K_{p,q}) = p - 1$ if $p \geq 3$ and $q \geq 4$. ■

We next present a characterization of the class of $\gamma_d$-strongly stable graphs, that is, the graphs for which for every subset $E' \subseteq E$, either $\gamma_d(G - E') = \gamma_d(G)$ or $\delta(G - E') = 0$. We need the following lemma.

**Lemma 14** *Every graph $G$ contains a spanning subgraph $G'$, in which every vertex is a leaf or a support vertex.*

**Proof.** Assume that some vertex of $G$ is neither a leaf nor a support vertex. Let $e_1$ be an edge of $G$ which is not incident to any leaf. Let $G_1 = G - e_1$. If every vertex of $G_1$ is a leaf or a support vertex, then let $G' = G_1$; otherwise let $G_2$ be a graph obtained from $G_1$ by removing an edge non-incident to any leaf. Let us observe that after a finite number of analogical steps we get a graph $G' = G_k$ every vertex of which is a leaf or a support vertex. ■

**Theorem 15** *For every graph $G$, the following conditions are equivalent:*

- $\gamma_d(G) = n$;
- *every vertex of $G$ is a leaf or a support vertex;*
- $b_d(G) = 0$.

**Proof.** Harary and Haynes [10] proved that the first two conditions are equivalent. We prove that the last two are also equivalent.

If every vertex of $G$ is a leaf or a support vertex, then Observations 1 and 2 imply that $\gamma_d(G) = n$. We have $b_d(G) = 0$ as the double domination number cannot be increased.

Now assume that some vertex of $G$, say $x$, is neither a leaf nor a support vertex. Thus $x$ has at least two neighbors. Moreover, each one of these neighbors has a neighbor other than $x$. It is not difficult to observe that

$V(G)\setminus\{x\}$ is a DDS of the graph $G$. Therefore $\gamma_d(G) \leq n-1$. By Lemma 14, the graph $G$ has a spanning subgraph $G'$, in which every vertex is a leaf or a support vertex. We have $\gamma_d(G') = n$. This implies that $b_d(G) > 0$. ■

A paired dominating set of a graph $G$ is a dominating set of vertices whose induced subgraph has a perfect matching. The paired domination number of $G$, denoted by $\gamma_{pr}(G)$, is the minimum cardinality of a paired dominating set of $G$. The paired bondage number, denoted by $b_{pr}(G)$, is the minimum cardinality among all sets of edges $E' \subseteq E$ such that $\delta(G - E') \geq 1$ and $\gamma_{pr}(G - E') > \gamma_{pr}(G)$. If for every $E' \subseteq E$, either $\gamma_{pr}(G - E') = \gamma_{pr}(G)$ or $\delta(G - E') = 0$, then we define $b_{pr}(G) = 0$, and we say that $G$ is a $\gamma_{pr}$-strongly stable graph. Raczek [23] observed that if $H \subseteq G$, then $b_{pr}(H) \leq b_{pr}(G)$. Let us observe that no result of such type is possible for the double bondage. Consider the complete bipartite graphs $K_{3,3}$, $K_{3,5}$, and $K_{4,5}$. Obviously, $K_{3,3} \subseteq K_{3,5} \subseteq K_{4,5}$. Using Proposition 13 we get $b_d(K_{3,3}) = 3 > 2 = b_d(K_{3,5}) < 3 = b_d(K_{4,5})$.

# 3 A characterization of trees with $b_d(T) = 1$

The authors of [7] proved that the bondage number of any tree is either one or two. In [26] it is observed that for any non-negative integer there exists a tree such that its double bondage number equals that number. The double bondage number of the subdivided star obtained from $K_{1,m}$ is $m-1$.

Hartnell and Rall [12] characterized all trees with $b(T) = 2$. The trees with $b_d(T) = 0$ are characterized in Theorem 15. In this section we characterize all trees with $b_d(T) = 1$.

First we need to define a tree $G_1$, see Figure 1.

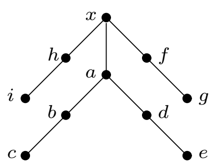


Figure 1: The tree $G_1$

Let $\mathcal{T}_0$ be a family of trees, not containing $P_4$, having a vertex adjacent to a path $P_3$, or a vertex adjacent to the tree $G_1$ through the vertex $x$ and to a leaf or a support vertex.

**Lemma 16** *If $T \in \mathcal{T}_0$, then $b_d(T) = 1$.*

**Proof.** First assume that some vertex of $T$, say $y$, is adjacent to a path $P_3$, say $v_1v_2v_3$. Let $y$ and $v_1$ be adjacent. Suppose that $b_d(T) \neq 1$. Thus for every edge $e$ of $T$ we have $\gamma_d(T-e) = \gamma_d(T)$. Let $T' = T - v_2 - v_3$ and $T'' = T' - v_1$. Since $P_4 \notin \mathcal{T}_0$, removing the edge $yv_1$ does not produce an isolated vertex. We have $\gamma_d(T) = \gamma_d(T - yv_1) = \gamma_d(T'' \cup P_3) = \gamma_d(T'')$ $+\gamma_d(P_3) = \gamma_d(T'') + 3$. We also have $\gamma_d(T) = \gamma_d(T - v_1v_2) = \gamma_d(T' \cup P_2)$ $= \gamma_d(T')+\gamma_d(P_2) = \gamma_d(T')+2$. Now we get $\gamma_d(T') = \gamma_d(T)-2 = \gamma_d(T'')+1$. This implies that there exists a $\gamma_d(T'')$-set that contains the vertex $y$. Let $D''$ be such a set. It is easy to observe that $D'' \cup \{v_2, v_3\}$ is a DDS of the tree $T$. Thus $\gamma_d(T) \leq \gamma_d(T'') + 2$, which contradicts $\gamma_d(T) = \gamma_d(T'') + 3$. Therefore $b_d(T) = 1$.

Now assume that some support vertex of $T$, say $y$, is adjacent to a tree $G_1$ through the vertex $x$. Let $z$ be a leaf adjacent to $y$. Let $T' = T-b-c$. Let $D'$ be any $\gamma_d(T')$-set. By Observations 1 and 2 we have $e, g, z, i, d, f, y, h \in D'$. Some of the vertices $a$ and $x$ belongs to the set $D'$ as the vertex $a$ has to be dominated twice. Without loss of generality we assume that $a \in D'$. Let us observe that $D' \setminus \{a\} \cup \{b, c\}$ is DDS of the tree $T$. Therefore $\gamma_d(T) \leq \gamma_d(T')+1$. Now we get $\gamma_d(T-ab) = \gamma_d(T' \cup P_2) = \gamma_d(T')+\gamma_d(P_2)$ $= \gamma_d(T') + 2 \geq \gamma_d(T) + 1 > \gamma_d(T)$. This implies that $b_d(T) = 1$.

Now assume that some vertex of $T$, say $y$, is adjacent to a support vertex, say $z$, and to a tree $G_1$ through the vertex $x$. Let $k$ be a leaf adjacent to $z$. If $\gamma_d(T - ab) > \gamma_d(T)$, then $b_d(T) = 1$. Now assume that $\gamma_d(T - ab) = \gamma_d(T)$. Let $T' = T - b - c$. We get $\gamma_d(T) = \gamma_d(T - ab)$ $= \gamma_d(T' \cup P_2) = \gamma_d(T') + \gamma_d(P_2) = \gamma_d(T') + 2$. Let us observe that there exists a $\gamma_d(T')$-set that does not contain the vertex $a$. Let $D'$ be such a set. The vertex $a$ has to be dominated twice, thus $x \in D'$. Obviously, $D' \cup \{b, c\}$ is a DDS of the tree $T$. We have $|D' \cup \{b, c\}| = |D'|+2 = \gamma_d(T')+2$ $= \gamma_d(T)$. This implies that $D' \cup \{b, c\}$ is a $\gamma_d(T)$-set, which contains the vertex $x$. Now let $D$ be any $\gamma_d(T)$-set that contains $x$. By Observations 1 and 2 we have $b, d, f, g, h, i, k, z \in D$. Let us observe that the set $D$ does not contain neither the vertex $y$ nor its any neighbor other than $x$ and $z$. Otherwise $D \setminus \{x\}$ is a DDS of the tree $T$, a contradiction to the minimality of $D$. Let $v_1 = x, v_2, \ldots, v_{d_T(y)-1}$ be the neighbors of $y$ other than $z$. Let $T_i$ denote the component of $T - yv_i$, which contains the vertex $v_i$. By $T''$ we denote the component of $T - yz$, which contains $y$. Let us observe that no $\gamma_d(T_i)$-set contains the vertex $x$, since the set $D$ contains the vertex $x$ and it is minimal. Consequently, no $\gamma_d(T)$-set contains more than two vertices from the closed neighborhood of the vertex $y$. Therefore there does not exist a $\gamma_d(T)$-set $D$ such that $D \cap V(T'')$ is a DDS of the tree $T''$. This implies that $\gamma_d(T'') > |D \cap V(T'')|$. Let us observe that $\gamma_d(T-T'') = |D \cap V(T-T'')|$ as $z$ is also a support vertex in the tree $T-T''$. Now we get $\gamma_d(T - yz) = \gamma_d(T'' \cup (T - T'')) = \gamma_d(T'') + \gamma_d(T - T'')$ $= \gamma_d(T'')+|D \cap V(T-T'')| > |D \cap V(T'')|+|D \cap V(T-T'')| = |D| = \gamma_d(T)$.

We conclude that $b_d(T) = 1$. ■

We characterize all trees with the double bondage number equaling one. For this purpose we introduce a family of trees $\mathcal{T}$ which contains all trees of the family $\mathcal{T}_0$, and trees that can be obtained as follows. Let $T_1$ be an element of $\mathcal{T}_0$. If $k$ is a positive integer, then $T_{k+1}$ can be obtained recursively from $T_k$ by one of the following operations.

- Operation $\mathcal{O}_1$: Attach a vertex by joining it to a support vertex of $T_k$.
- Operation $\mathcal{O}_2$: Attach a path $P_2$ by joining one of its vertices to a vertex of $T_k$, which is a support vertex, or is adjacent to a support vertex and to another vertex which has degree two.
- Operation $\mathcal{O}_3$: Attach a subdivided star by joining the vertex of minimum eccentricity to any vertex of $T_k$.

Now we prove that the double bondage number of every tree of the family $\mathcal{T}$ is one.

**Lemma 17** *If $T \in \mathcal{T}$, then $b_d(T) = 1$.*

**Proof.** If $T \in \mathcal{T}_0$, then by Lemma 16 we have $b_d(T) = 1$. Now assume that $T \in \mathcal{T} \setminus \mathcal{T}_0$. We use the induction on the number $k$ of operations performed to construct the tree $T$. Let $k \geq 2$ be an integer. Assume that the result is true for every tree $T' = T_k$ of the family $\mathcal{T}$ constructed by $k-1$ operations. Let $T = T_{k+1}$ be a tree of the family $\mathcal{T}$ constructed by $k$ operations.

First assume that $T$ is obtained from $T'$ by operation $\mathcal{O}_1$. Let $x$ be the attached vertex, and let $y$ be its neighbor. Let $z$ be a leaf adjacent to $y$ and different from $x$. Let $D'$ be any $\gamma_d(T')$-set. By Observation 2 we have $y \in D'$. It is easy to observe that $D' \cup \{x\}$ is a DDS of the tree $T$. Thus $\gamma_d(T) \leq \gamma_d(T') + 1$. The assumption $b_d(T') = 1$ implies that there is an edge $e$ of $T'$ such that $\gamma_d(T' - e) > \gamma_d(T')$. Since $z$ is a leaf of $T'$, we have $e \neq yz$. By $T_y$ ($T'_y$, respectively) we denote the component of $T - e$ ($T' - e$, respectively) which contains the vertex $y$. Let $D_y$ be any $\gamma_d(T_y)$-set. By Observations 1 and 2 we have $x, y, z \in D_y$. It is easy to observe that $D_y \setminus \{x\}$ is a DDS of the tree $T'_y$. Therefore $\gamma_d(T'_y) \leq \gamma_d(T_y) - 1$. Now we get $\gamma_d(T - e) = \gamma_d(T - e - T_y) + \gamma_d(T_y) \geq \gamma_d(T - e - T_y) + \gamma_d(T'_y) + 1$ $= \gamma_d(T' - e - T'_y) + \gamma_d(T'_y) + 1 = \gamma_d(T' - e) + 1 > \gamma_d(T') + 1 \geq \gamma_d(T)$. This implies that $b_d(T) = 1$.

Now assume that $T$ is obtained from $T'$ by operation $\mathcal{O}_2$. The vertex to which is attached $P_2$ we denote by $x$. Let $v_1v_2$ denote the attached path. Let $v_1$ be joined to $x$. Let $D'$ be any $\gamma_d(T')$-set. Obviously, $D' \cup \{v_1, v_2\}$ is a DDS of the tree $T$. Thus $\gamma_d(T) \leq \gamma_d(T') + 2$. If $x$ is adjacent to a leaf, then we denote it by $y$. The assumption $b_d(T') = 1$ implies that there is an edge $e$ of $T'$ such that $\gamma_d(T'-e) > \gamma_d(T')$. Since $y$ is a leaf of $T'$, we have $e \neq xy$. By $T_x$ ($T'_x$, respectively) we denote the component of $T-e$ ($T'-e$, respectively) which contains the vertex $x$. Let $D_x$ be any $\gamma_d(T_x)$-set. By Observations 1 and 2 we have $v_1, v_2, x, y \in D_x$. It is easy to observe that $D_x \setminus \{v_1, v_2\}$ is a DDS of the tree $T'_x$. Therefore $\gamma_d(T'_x) \leq \gamma_d(T_x) - 2$. Now assume that $x$ is adjacent to a support vertex, say $y$, and to a vertex of degree two other than $y$, say $z$. The assumption $b_d(T') = 1$ implies that there is an edge $e$ of $T'$ such that $\gamma_d(T'-e) > \gamma_d(T')$. By $T_x$ ($T'_x$, respectively) we denote the component of $T-e$ ($T'-e$, respectively) which contains the vertex $x$. Let $D$ be any $\gamma_d(T_x)$-set. By Observations 1 and 2 we have $v_1, v_2, y \in D_x$. At least one of the vertices $x$ and $z$ belongs to the set $D_x$ as the vertex $z$ has to be dominated twice. Let us observe that $D_x \setminus \{v_1, v_2\}$ is a DDS of the tree $T'_x$ as the vertex $x$ is still dominated at least twice. Now we conclude that $\gamma_d(T'_x) \leq \gamma_d(T_x) - 2$. We get $\gamma_d(T-e) = \gamma_d(T-e-T_x) + \gamma_d(T_x) \geq \gamma_d(T-e-T_x) + \gamma_d(T'_x) + 2$ $= \gamma_d(T'-e-T'_x) + \gamma_d(T'_x) + 2 = \gamma_d(T'-e) + 2 > \gamma_d(T') + 2 \geq \gamma_d(T)$. This implies that $b_d(T) = 1$.

Now assume that $T$ is obtained from $T'$ by operation $\mathcal{O}_3$. Let $x$ be the vertex to which is attached the subdivided star, and let $y$ denote its central vertex. Let $D'$ be any $\gamma_d(T')$-set. It is easy to observe that $D' \cup V(T-T') \setminus \{y\}$ is a DDS of the tree $T$. Thus $\gamma_d(T) \leq \gamma_d(T') + 2d_T(y) - 2$. The assumption $b_d(T') = 1$ implies that there is an edge $e$ of $T'$ such that $\gamma_d(T'-e) > \gamma_d(T')$. By $T_x$ ($T'_x$, respectively) we denote the component of $T-e$ ($T'-e$, respectively) which contains the vertex $x$. Let us observe that there exists a $\gamma_d(T_x)$-set that does not contain the vertex $y$. Let $D_x$ be such a set. Observations 1 and 2 imply that $V(T-T') \setminus \{y\} \subseteq D_x$. Observe that $D_x \cap V(T')$ is a DDS of the tree $T'_x$. Therefore $\gamma_d(T'_x) \leq \gamma_d(T_x) - 2d_T(y) + 2$. Now we get $\gamma_d(T-e) = \gamma_d(T-e-T_x) + \gamma_d(T_x) \geq \gamma_d(T-e-T_x) + \gamma_d(T'_x)$ $+ 2d_T(y) - 2 = \gamma_d(T'-e-T'_x) + \gamma_d(T'_x) + 2d_T(y) - 2 = \gamma_d(T'-e) + 2d_T(y) - 2$ $> \gamma_d(T') + 2d_T(y) - 2 \geq \gamma_d(T)$. This implies that $b_d(T) = 1$. ■

Now we prove that if the double bondage number of a tree equals one, then the tree belongs to the family $\mathcal{T}$.

**Lemma 18** *Let $T$ be a tree. If $b_d(T) = 1$, then $T \in \mathcal{T}$.*

**Proof.** Let $n$ mean the number of vertices of the tree $T$. We proceed by induction on this number. If $\mathrm{diam}(T) \leq 3$, then Observations 1 and 2 imply that $\gamma_d(T) = n$ as every vertex of $T$ is a leaf or a support vertex. We have $b_d(T) = 0$ as the double bondage number cannot be increased.

Now assume that $\mathrm{diam}(T) \geq 4$. The result we obtain by the induction on the number $n$. Assume that the lemma is true for every tree $T'$ of order $n' < n$.

First assume that some support vertex of $T$, say $x$, is strong. Let $y$ and $z$ be leaves adjacent to $x$. Let $T' = T - y$. Let $D$ be any $\gamma_d(T)$-set. By Observations 1 and 2 we have $x, y, z \in D$. It is easy to observe that $D \setminus \{y\}$ is a DDS of the tree $T'$. Therefore $\gamma_d(T') \leq \gamma_d(T) - 1$. The assumption $b_d(T) = 1$ implies that there is an edge $e$ of $T$ such that $\gamma_d(T - e) > \gamma_d(T)$. Since $y$ is a leaf of $T$, we have $e \neq xy$. Consequently, $e \in E(T')$. By $T_x$ ($T'_x$, respectively) we denote the component of $T - e$ ($T' - e$, respectively) which contains the vertex $x$. Let $D'_x$ be any $\gamma_d(T'_x)$-set. By Observation 2 we have $x \in D'_x$. It is easy to observe that $D'_x \cup \{y\}$ is a DDS of the tree $T_x$. Thus $\gamma_d(T_x) \leq \gamma_d(T'_x) + 1$. Now we get $\gamma_d(T' - e) = \gamma_d(T' - e - T'_x) + \gamma_d(T'_x) \geq \gamma_d(T' - e - T'_x) + \gamma_d(T_x) - 1$ $= \gamma_d(T - e - T_x) + \gamma_d(T_x) - 1 = \gamma_d(T - e) - 1 > \gamma_d(T) - 1 \geq \gamma_d(T')$. This implies that $b_d(T') = 1$. By the inductive hypothesis we have $T' \in \mathcal{T}$. The tree $T$ can be obtained from $T'$ by operation $\mathcal{O}_1$. Thus $T \in \mathcal{T}$. Henceforth, we can assume that every support vertex of $T$ is weak.

We now root $T$ at a vertex $r$ of maximum eccentricity $\mathrm{diam}(T)$. Let $t$ be a leaf at maximum distance from $r$, $v$ be the parent of $t$, $u$ be the parent of $v$, and $w$ be the parent of $u$ in the rooted tree. If $\mathrm{diam}(T) \geq 5$, then let $d$ denote the parent of $w$. By $T_x$ let us denote the subtree induced by a vertex $x$ and its descendants in the rooted tree $T$.

First assume that $d_T(u) = 2$. If $d_T(w) = 1$, then $T = P_4$. We have $b_d(T) = 0$, a contradiction. Now assume that $d_T(w) \geq 2$. The vertex $w$ is adjacent to a path $P_3$ in $T \neq P_4$, thus $T \in \mathcal{T}_0 \subseteq \mathcal{T}$.

Now assume that $d_T(u) \geq 3$. Assume that some child of $u$, say $x$, is a leaf. Let $T' = T - T_v$. Let $D$ be any $\gamma_d(T)$-set. By Observations 1 and 2 we have $t, x, v, u \in D$. It is easy to observe that $D \setminus \{v, t\}$ is a DDS of the tree $T'$. Therefore $\gamma_d(T') \leq \gamma_d(T) - 2$. The assumption $b_d(T) = 1$ implies that there is an edge $e$ of $T$ such that $\gamma_d(T - e) > \gamma_d(T)$. Since the vertex $t$ is a leaf of the tree $T$, we have $e \neq vt$. Let us observe that $e \neq uv$. We have $\gamma_d(T - uv) = \gamma_d(T' \cup P_2) = \gamma_d(T') + 2 \leq \gamma_d(T)$. Therefore $e \neq uv$. Consequently, $e \in E(T')$. By $T_u$ ($T'_u$, respectively) we denote the component of $T - e$ ($T' - e$, respectively) which contains the vertex $u$. Let $D'_u$ be any $\gamma_d(T'_u)$-set. Obviously, $D'_u \cup \{v, t\}$ is a DDS of the tree $T_u$. Thus $\gamma_d(T_u) \leq \gamma_d(T'_u) + 2$. Now we get $\gamma_d(T' - e) = \gamma_d(T' - e - T'_u) + \gamma_d(T'_u)$ $\geq \gamma_d(T' - e - T'_u) + \gamma_d(T_u) - 2 = \gamma_d(T - e - T_u) + \gamma_d(T_u) - 2 = \gamma_d(T - e) - 2$ $> \gamma_d(T) - 2 \geq \gamma_d(T')$. This implies that $b_d(T') = 1$. By the inductive hypothesis we have $T' \in \mathcal{T}$. The tree $T$ can be obtained from $T'$ by operation $\mathcal{O}_2$. Thus $T \in \mathcal{T}$.

Thus we can assume that all children of $u$ are support vertices of degree two. Let $x$ be a child of $u$ other than $v$. The leaf adjacent to $x$ we denote by $y$. First assume that $d_T(u) \geq 4$. Let $k$ be a child of $u$ other than $v$ and $x$. The leaf adjacent to $k$ we denote by $l$. Let $T' = T - T_u$. Let us observe that there exists a $\gamma_d(T)$-set that does not contain the vertex $u$. Let $D$ be such a set. Observations 1 and 2 imply that $V(T_u)\setminus\{u\} \subseteq D$. Observe that $D \cap V(T')$ is a DDS of the tree $T'$. Therefore $\gamma_d(T') \le \gamma_d(T) - 2d_T(u) + 2$. The assumption $b_d(T) = 1$ implies that there is an edge $e$ of $T$ such that $\gamma_d(T-e) > \gamma_d(T)$. Let us observe that $e$ is not incident to the vertex $u$ and a child of $u$. Suppose otherwise. Without loss of generality we assume that $e = uv$. Let $T'' = T - T_v$. Let $D$ be any $\gamma_d(T)$-set. By Observations 1 and 2 we have $t, v, x, k \in D$. Let us observe that $D\setminus\{v, t\}$ is a DDS of the tree $T''$. Therefore $\gamma_d(T'') \le \gamma_d(T) - 2$. Now we get $\gamma_d(T - uv) = \gamma_d(T'' \cup P_2)$ $= \gamma_d(T'') + 2 \le \gamma_d(T)$, a contradiction. Therefore $e$ is not an edge between the vertex $u$ and a child of $u$. The other edges incident to the children of $u$ are incident to leaves. Therefore $e$ is not incident to any child of $u$. We have $\gamma_d(T - wu) = \gamma_d(T' \cup T_u) = \gamma_d(T') + \gamma_d(T_u) = \gamma_d(T') + 2d_T(u) - 2$ $\le \gamma_d(T)$. Therefore $e \ne wu$. Now we conclude that $e \in E(T')$. By $T_w$ ($T'_w$, respectively) we denote the component of $T - e$ ($T' - e$, respectively) which contains the vertex $w$. Let $D'_w$ be any $\gamma_d(T'_w)$-set. It is easy to observe that $D'_w \cup V(T_u) \setminus \{u\}$ is a DDS of the tree $T_w$. Thus $\gamma_d(T_w)$ $\le \gamma_d(T'_w) + 2d_T(u) - 2$. Now we get $\gamma_d(T' - e) = \gamma_d(T' - e - T'_w) + \gamma_d(T'_w)$ $\ge \gamma_d(T' - e - T'_w) + \gamma_d(T_w) - 2d_T(u) + 2 = \gamma_d(T - e - T_w) + \gamma_d(T_w)$ $-2d_T(u) + 2 = \gamma_d(T - e) - 2d_T(u) + 2 > \gamma_d(T) - 2d_T(u) + 2 \ge \gamma_d(T')$. This implies that $b_d(T') = 1$. By the inductive hypothesis we have $T' \in \mathcal{T}$. The tree $T$ can be obtained from $T'$ by operation $\mathcal{O}_3$. Thus $T \in \mathcal{T}$.

Now assume that $d_T(u) = 3$. First assume that $d_T(w) = 2$, or there is a child of $w$ other than $u$, say $k$, such that the distance of $w$ to the most distant vertex of $T_k$ is three or one. Then it suffices to consider the possibilities when $T_k$ is isomorphic to $T_u$, or $k$ is a leaf. Let $T' = T - T_u$. Let us observe that there exists a $\gamma_d(T)$-set that does not contain the vertex $u$. Let $D$ be such a set. By Observations 1 and 2 we have $t, y, v, x \in D$. Observe that $D \setminus \{v, t, x, y\}$ is a DDS of the tree $T'$. Therefore $\gamma_d(T')$ $\le \gamma_d(T) - 4$. The assumption $b_d(T) = 1$ implies that there is an edge $e$ of $T$ such that $\gamma_d(T - e) > \gamma_d(T)$. If $T_k$ is isomorphic to $T_u$, then without loss of generality we may assume that $e \notin E(T_u) \cup \{wu\}$. Now consider the cases when $d_T(w) = 2$ or $k$ is a leaf. We have $e \ne vt, xy$ as the vertices $t$ and $y$ are leaves of the tree $T$. Let us observe that $e$ is not incident to $u$. We have $\gamma_d(T - wu) = \gamma_d(T' \cup P_5) = \gamma_d(T') + \gamma_d(P_5) = \gamma_d(T') + 4 \le \gamma_d(T)$. Now suppose that $e \in \{uv, ux\}$. Without loss of generality we may assume that $e = uv$. Let $T'' = T - T_v$. Let $D$ be a $\gamma_d(T)$-set that does not contain the vertex $u$. By Observations 1 and 2 we have $t, v, x \in D$. If $k$ is a leaf, then Observation 2 implies that $w \in D$. If $d_T(w) = 2$, then also the vertex $w$ belongs to the set $D$ as it has to be dominated twice. Let us observe that $D \setminus \{v, t\}$ is a DDS of the tree $T''$. Therefore $\gamma_d(T'') \le \gamma_d(T) - 2$. Now we get $\gamma_d(T - uv) = \gamma_d(T'' \cup P_2) = \gamma_d(T'') + \gamma_d(P_2) = \gamma_d(T'') + 2 \le \gamma_d(T)$, a contradiction. We conclude that $e \in E(T')$. By $T_w$ ($T'_w$, respectively) we denote the component of $T - e$ ($T' - e$, respectively) which contains the vertex $w$. Let $D'_w$ be any $\gamma_d(T'_w)$-set. It is easy to observe that $D'_w \cup \{v,t,x,y\}$ is a DDS of the tree $T_w$. Thus $\gamma_d(T_w) \le \gamma_d(T'_w)+4$. Now we get $\gamma_d(T'-e) = \gamma_d(T'-e-T'_w) + \gamma_d(T'_w) \ge \gamma_d(T'-e-T'_w) + \gamma_d(T_u) - 4 = \gamma_d(T-e-T_w) + \gamma_d(T_w) - 4 = \gamma_d(T-e) - 4 > \gamma_d(T) - 4 \ge \gamma_d(T')$. This implies that $b_d(T') = 1$. By the inductive hypothesis we have $T' \in \mathcal{T}$. The tree $T$ can be obtained from $T'$ by operation $\mathcal{O}_3$. Thus $T \in \mathcal{T}$.

Now assume that every child of $w$ other than $u$ is a support vertex of degree two. Let $k$ be a child of $w$ other than $u$. The leaf adjacent to $k$ we denote by $l$. First assume that $d_T(w) \ge 5$. Let $m$ and $q$ be children of $w$ other than $u$ and $k$. Let $T' = T - T_k$. Let $D$ be any $\gamma_d(T)$-set. By Observations 1 and 2 we have $k,l,m,q \in D$. It is easy to observe that $D \setminus \{k,l\}$ is a DDS of the tree $T'$. Therefore $\gamma_d(T') \le \gamma_d(T) - 2$. The assumption $b_d(T) = 1$ implies that there is an edge $e$ of $T$ such that $\gamma_d(T-e) > \gamma_d(T)$. Since $l$ is a leaf of $T$, we have $e \ne kl$. Let us observe that $e \ne wk$. We have $\gamma_d(T-wk) = \gamma_d(T' \cup P_2) = \gamma_d(T') + \gamma_d(P_2) = \gamma_d(T')+2 \le \gamma_d(T)$. Now we conclude that $e \in E(T')$. By $T_w$ ($T'_w$, respectively) we denote the component of $T-e$ ($T'-e$, respectively) which contains the vertex $w$. Let $D'_w$ be any $\gamma_d(T'_w)$-set. Obviously, $D'_w \cup \{k,l\}$ is a DDS of the tree $T_w$. Thus $\gamma_d(T_w) \le \gamma_d(T'_w)+2$. Now we get $\gamma_d(T'-e) = \gamma_d(T'-e-T'_w) + \gamma_d(T'_w) \ge \gamma_d(T'-e-T'_w) + \gamma_d(T_w) - 2 = \gamma_d(T-e-T_w) + \gamma_d(T_w) - 2 = \gamma_d(T-e) - 2 > \gamma_d(T) - 2 \ge \gamma_d(T')$. This implies that $b_d(T') = 1$. By the inductive hypothesis we have $T' \in \mathcal{T}$. The tree $T$ can be obtained from $T'$ by operation $\mathcal{O}_2$. Thus $T \in \mathcal{T}$.

Now assume that $d_T(w) = 3$. Let $T' = T - T_u$. The vertex $d$ is adjacent to a path $P_3$ in $T' \ne P_4$, thus $T' \in \mathcal{T}_0 \subseteq \mathcal{T}$. The tree $T$ can be obtained from $T'$ by operation $\mathcal{O}_3$. Thus $T \in \mathcal{T}$.

Now assume that $d_T(w) = 4$. Let $m$ be the child of $w$ other than $u$ and $k$. The leaf adjacent to $m$ we denote by $p$. First assume that there is a child of $d$ other than $w$, say $a$, such that the distance of $d$ to the most distant vertex of $T_a$ is four. It suffices to consider only the possibility when $T_a$ is isomorphic to $T_w$. Let $T' = T - T_u$. Similarly as earlier we get $\gamma_d(T') \le \gamma_d(T) - 4$. The assumption $b_d(T) = 1$ implies that there is an edge $e$ of $T$ such that $\gamma_d(T-e) > \gamma_d(T)$. Because of the similarity between the subtrees $T_w$ and $T_a$, without loss of generality we assume that $e \notin E(T_w)$. Thus $e \in E(T')$. By $T_w$ ($T'_w$, respectively) we denote the component of $T-e$ ($T'-e$, respectively) which contains the vertex $w$. Let $D'_w$ be any $\gamma_d(T'_w)$-set. It is easy to observe that $D'_w \cup \{v,t,x,y\}$ is a DDS of the tree $T_w$. Thus $\gamma_d(T_w) \le \gamma_d(T'_w)+4$. Now we get $\gamma_d(T'-e) = \gamma_d(T'-e-T'_w) + \gamma_d(T'_w) \ge \gamma_d(T'-e-T'_w) + \gamma_d(T_w) - 4 = \gamma_d(T-e-T_w) + \gamma_d(T_w) - 4 = \gamma_d(T-e) - 4 > \gamma_d(T) - 4 \ge \gamma_d(T')$. This implies that $b_d(T') = 1$. By the inductive hypothesis we have $T' \in \mathcal{T}$. The tree $T$ can be obtained from $T'$ by operation $\mathcal{O}_3$. Thus $T \in \mathcal{T}$.

Now assume that there is a child of $d$, say $a$, such that the distance of $d$

to the most distant vertex of $T_a$ is three. It suffices to consider only the possibility when $T_a$ is isomorphic to $T_u$. Let $b$ and $q$ denote the children of $a$. The leaf adjacent to $b$ we denote by $c$, and the leaf adjacent to $q$ we denote by $z$. Let $T' = T - T_a$. Let us observe that there exists a $\gamma_d(T)$-set that does not contain the vertex $a$. Let $D$ be such a set. By Observations 1 and 2 we have $b, c, q, z \in D$. Observe that $D \setminus \{b, c, q, z\}$ is a DDS of the tree $T'$. Therefore $\gamma_d(T') \leq \gamma_d(T) - 4$. The assumption $b_d(T) = 1$ implies that there is an edge $e$ of $T$ such that $\gamma_d(T - e) > \gamma_d(T)$. First assume that $e \in E(T - T')$. Let $T''$ be a tree obtained from $T$ by replacing the edge $uw$ with $ua$. Let us observe that there exists a $\gamma_d(T'' - e)$-set that does not contain the vertex $u$. Let $D''$ be such a set. It is easy to observe that $D''$ is a DDS of the tree $T - e$. Therefore $\gamma_d(T - e) \leq \gamma_d(T'' - e)$. Similarly we conclude that $\gamma_d(T'' - e) \leq \gamma_d(T - e)$. This implies that $\gamma_d(T - e) = \gamma_d(T'' - e)$. Let $e'$ be an edge of $T_w - T_u$ corresponding to $e$ in the subtree $T_a$. Let us observe that the graphs $T'' - e$ and $T - e'$ are isomorphic. Therefore $\gamma_d(T'' - e) = \gamma_d(T - e')$, and consequently, $\gamma_d(T - e') = \gamma_d(T - e)$. This implies that we may assume that $e \in E(T')$. By $T_d$ ($T'_d$, respectively) we denote the component of $T - e$ ($T' - e$, respectively) which contains the vertex $d$. Let $D'_d$ be any $\gamma_d(T'_d)$-set. It is easy to observe that $D'_d \cup \{b, c, q, z\}$ is a DDS of the tree $T_d$. Thus $\gamma_d(T_d) \leq \gamma_d(T'_d) + 4$. Now we get $\gamma_d(T' - e) = \gamma_d(T' - e - T'_d) + \gamma_d(T'_d) \geq \gamma_d(T' - e - T'_d) + \gamma_d(T_d) - 4$ $= \gamma_d(T - e - T_d) + \gamma_d(T_d) - 4 = \gamma_d(T - e) - 4 > \gamma_d(T) - 4 \geq \gamma_d(T')$. This implies that $b_d(T') = 1$. By the inductive hypothesis we have $T' \in \mathcal{T}$. The tree $T$ can be obtained from $T'$ by operation $\mathcal{O}_3$. Thus $T \in \mathcal{T}$.

Now assume that there is a child of $d$, say $a$, such that the distance of $d$ to the most distant vertex of $T_a$ is two or one. Thus $a$ is a support vertex or a leaf. Let us observe that $d$ is adjacent to a tree $G_1$ through the vertex $x$ and to a support vertex or a leaf. Thus $T \in \mathcal{T}_0 \subseteq \mathcal{T}$.

Now assume that $d_T(d) = 2$. Let $T' = T - T_k$. Let $D$ be any $\gamma_d(T)$-set. By Observations 1 and 2 we have $k, l, m \in D$. Some of the vertices $d$ and $w$ belongs to the set $D$ as the vertex $d$ has to be dominated twice. Let us observe that $D \setminus \{k, l\}$ is a DDS of the tree $T'$. Therefore $\gamma_d(T') \leq \gamma_d(T) - 2$. Similarly as when $d_T(w) \geq 5$, we conclude that $b_d(T') = 1$. By the inductive hypothesis we have $T' \in \mathcal{T}$. The tree $T$ can be obtained from $T'$ by operation $\mathcal{O}_2$. Thus $T \in \mathcal{T}$. ∎

As an immediate consequence of Lemmas 17 and 18, we have the following characterization of trees with the double bondage number equaling one.

**Theorem 19** *Let $T$ be a tree. Then $b_d(T) = 1$ if and only if $T \in \mathcal{T}$.*

It is an open problem to characterize all graphs with the double bondage number equaling one.

**Problem 20** *Characterize graphs $G$ such that $b_d(G) = 1$.*

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